Numerical Analysis: Solutions of System of Linear Equation

Natasha S. Sharma, PhD

Mathematical Question we are interested in answering numerically

- How to solve the following linear system for x

$$Ax = b?$$

where A is an $n \times n$ invertible matrix and b is vector of length n.
Notation: x^* denote the true solution to $Ax = b$.

- Traditional Approaches:
  1. Gaussian Elimination with backward substitution/ row-echelon form.
  2. LU Decomposition of A and solving two smaller linear systems.
- Goal: Numerically approximate x^* by $\{x_n\}_{n \geq 1}$ based on an initial guess x_0 such that

$$x_n \to x^* \text{ as } n \to \infty.$$

Why do we need to approximate the solution?

1. Gaussian Elimination and LU decomposition provide the exact solution under the assumption of infinite precision (an impractical assumption when solving large systems). We need to be able to design a method that takes into consideration this issue **(Residual Correction Method)**.
2. Computationally demanding to use direct methods to solve systems with $n \approx 10^6$ iterative methods need less memory for each solve.
3. Ill-conditioned systems that is, sensitivity of the solution to $Ax = b$, to a change in b.

Algorithmns for solving linear systems

1. **Direct Methods**: Gaussian Elimination, *LU* decomposition.
   They involve one large computational step.
2. **Iterative Methods:** Residual Correction Method, Jacobi Method Gauss-Seidel Method (scope of this course).
   Given error tolerance ε and an initial guess vector x_0, these methods approach the solution gradually.

The big advantage of the iterative methods their memory usage, which is significantly less than a direct solver for the same sized problems.

Iterative Methods: Residual Correction Mehtod

Motivation: To overcome the unavoidable round-off errors in solving linear systems.
Consider:

$$x - \frac{800}{801}y = 10$$
$$-x + y = 50.$$

Recall, the exact solution is $\mathbf{x}^* = [48010, 48060]^T$.
Assuming $800/801 \approx 0.998751560549313$,
the computed solution $x^{(0)}$ using three digits of significance is inaccurate.
Goal: **Predict** the error in the computed solution and **correct** the error.

Algorithmns for solving linear systems

Definition (Residual)

Residual

$$r = b - A\hat{x}$$

where $\hat{x}$ is the imprecise computed solution.

Definition (Error)

Error

$$\hat{e} = x^* - \hat{x}$$

where x^* is the exact solution and $\hat{x}$ is the imprecise computed solution.

Residual Correction Method

Relation between residual and error

$$r = b - A\hat{x} = Ax^* - A\hat{x} = A(x^* - \hat{x}) = A\hat{e}.$$

This motivates the definition of the residual correction method where the corrected solution say x^c is given by

$$x^c = \hat{x} + \underbrace{\hat{e}}_{x^* - \hat{x}}$$

Residual Correction Method

Input: $x^0 = \hat{x}$ obtained from using Gauss Elimination to solve $Ax = b$.

Tolerance $\varepsilon > 0$

Let $r^0 = b - Ax^0$

Solve for e^0 satisfying $Ae^0 = r^0$.

while $|e^n| > \varepsilon$ **do**

  $x^{n+1} = x^n + e^n$

  Let $r^{n+1} = b - Ax^{n+1}$.

  Solve for e^{n+1} satisfying $Ae^{n+1} = r^{n+1}$.

end

Example

Using a computer with four-digit precision, employing Gaussian elimination solve the system

$$\begin{aligned} x_1 + 0.5x_2 + 0.3333x_3 &= 1 \\ 0.5x_1 + 0.3333x_2 + 0.25x_3 &= 0 \\ 0.3333x_1 + 0.25x_2 + 0.2x_3 &= 0. \end{aligned}$$

yields the solution $x^0 = [8.968, -35.77, 29.77]^T$.

Following the residual correction algorithm, taking $\varepsilon = 10^{-4}$ and given $x^0 = [8.968, -35.77, 29.77]^T$,

$$
\begin{aligned}
r^0 &= [-0.005341, -0.004359, -.0005344]^T \\
e^0 &= [0.09216, -0.5442, 0.5239]^T \\
x^1 &= [9.060, 36.31, 30.29]^T \\
r^1 &= [-0.000657, -0.000377, -.0000198]^T \\
e^1 &= [0.001707, -0.013, 0.0124]^T \\
x^1 &= [9.062, -36.32, 30.30]^T
\end{aligned}
$$

Jacobi Method/ Method of Simultaneous Replacements

Consider the following system

$$9x_1 + x_2 + x_3 = 10 \quad (1)$$

$$2x_1 + 10x_2 + 3x_3 = 19 \quad (2)$$

$$3x_1 + 4x_2 + 11x_3 = 0 \quad (3)$$

Given an initial guess $\mathbf{x}^{(0)} = [x_1^0, x_2^0, x_3^0]^T$, we construct a sequence based on the formula:

$$x_1^{(k+1)} = \frac{10 - x_2^{(k)} + x_3^{(k)}}{9}$$

$$x_2^{(k+1)} = \frac{19 - 2x_1^{(k)} - 3x_3^{(k)}}{10}$$

$$x_3^{(k+1)} = \frac{-(3x_1^{(k)} + 4x_2^{(k)})}{11}.$$

Performance of Jacobi Iteration

n	${x_1}^{(k)}$	${x_2}^{(k)}$	${x_3}^{(k)}$	Error	Ratio
0	0	0	0	2e+0	–
1	1.1111	1.9	0	1e+0	0.5
2	0.9	1.6778	-0.9939	3.22e-1	0.322
3	1.0351	2.0182	-0.8556	1.44e-1	0.448
4	0.9819	1.9496	-1.0162	-5.04e-2	0.349
5	1.0074	2.0085	-0.9768	2.32e-2	0.462
⋮	⋮	⋮	⋮	⋮	⋮
10	0.9999	1.9997	-1.003	2.8e-4	0.382
⋮	⋮	⋮	⋮	⋮	⋮
30	1	2	-1	3.011e-11	0.447
31	1	2	-1	1.35e-11	0.447

Gauss-Seidel Method/ Method of Successive Replacements

For the following system,

$$
\begin{aligned}
9x_1 + x_2 + x_3 &= 10 \\
2x_1 + 10x_2 + 3x_3 &= 19 \\
3x_1 + 4x_2 + 11x_3 &= 0
\end{aligned}
$$

Given an initial guess $\mathbf{x}^{(0)} = [x_1^0, x_2^0, x_3^0]^T$, we construct a sequence based on the Gauss-Seidel formula:

$$
\begin{aligned}
{x_1}^{(k+1)} &= \frac{10 - {x_2}^{(k)} + {x_3}^{(k)}}{9} \\
{x_2}^{(k+1)} &= \frac{19 - 2{x_1}^{(k+1)} - 3{x_3}^{(k)}}{10} \\
{x_3}^{(k+1)} &= \frac{-(3{x_1}^{(k+1)} + 4{x_2}^{(k+1)})}{11}.
\end{aligned}
$$

Performance of Gauss Seidel Iteration

n	${x_1}^{(k)}$	${x_2}^{(k)}$	${x_3}^{(k)}$	Error	Ratio
0	0	0	0	2e+0	–
1	1.1111	1.6778	-0.9131	3.22e-1	0.161
2	1.0262	1.9687	-0.9958	3.22e-1	0.097
3	1.003	1.9981	-1.0001	1.44e-1	0.096
4	1.002	2.000	-1.00001	-2.24e-4	0.074
5	1	2	-1	1.65e-5	0.074
6	1	2	-1	2.58e-6	0.155

General Schema: Towards Error Analysis

In order to examine the error analysis, we need to express the two iterative methods in a more compact form.
This can be achieved by expressing the formulas for Jacobi and Gauss Seidel using vector-matrix format.

Theorem (Vector-Matrix Format)

Every linear system

$$Ax = b$$

can be expressed in the form

$$Nx = b + Px, \quad A = N - P,$$

where N is an nonsingular (invertible) matrix. Furthermore, any iteration method can be described as:

$$N\mathbf{x}^{(k+1)} = b + P\mathbf{x}^{k}, \quad k = 0, 1, \cdots \tag{4}$$

Work Out Example

Example

Write the Jacobi method applied to the linear system (1)–(3) using the vector-matrix format (4):

$$N\mathbf{x}^{(k+1)} = b + P\mathbf{x}^k, \quad b = [10, 19, 0]^T.$$

$$x_1{}^{(k+1)} = \frac{10 - x_2{}^{(k)} + x_3{}^{(k)}}{9}$$

$$x_2{}^{(k+1)} = \frac{19 - 2x_1{}^{(k)} - 3x_3{}^{(k)}}{10}$$

$$x_3{}^{(k+1)} = \frac{-(3x_1{}^{(k)} + 4x_2{}^{(k)} + 11)}{11}.$$

Solution

1. Multiply first equation by 9, second by 10 and third by 11.

$$9{x_1}^{(k+1)} = 10 - {x_2}^{(k)} + {x_3}^{(k)}$$
$$10{x_2}^{(k+1)} = 19 - 2{x_1}^{(k)} - 3{x_3}^{(k)}$$
$$11{x_3}^{(k+1)} = 0 - (3{x_1}^{(k)} + 4{x_2}^{(k)}).$$

2. Recall we want to express the formula in the form $N\mathbf{x}^{(k+1)} = b + P\mathbf{x}^k, \quad b = [10, 19, 0]^T$.
   We already have b.
   We now need to derive the matrices N and P.
3. Obtain N first.

$$N\mathbf{x}^{(k+1)} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 11 \end{bmatrix} \begin{bmatrix} {x_1}^{(k+1)} \\ {x_2}^{(k+1)} \\ {x_3}^{(k+1)} \end{bmatrix} = \begin{bmatrix} 9{x_1}^{(k+1)} \\ 10{x_2}^{(k+1)} \\ 11{x_3}^{(k+1)} \end{bmatrix}$$

LHS of the linear system above!

Solution

$$P = \begin{bmatrix} 0 & -1 & 1 \\ -2 & 0 & -3 \\ -3 & 4 & 0 \end{bmatrix}, \quad N = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 11 \end{bmatrix}.$$

Observations

- For the Jacobi iterative method, the matrices N^J and P^J stay unchanged!
- Notice the zero diagonal entries for P.
- The diagonal entries of N and A are the same!
- Easy way to obtain P is

$$P = N - A$$

To summarize...

Since our next task is to extract the N and P characterizing the Gauss Seidel Method, we let N^J and P^J to denote the matrices charactering the vector-matrix format (4) for the Jacobi Iteration.
That is, For Jacobi Method solving $Ax = b$,

$$N^J \mathbf{x}^{(k+1)} = b + P^J \mathbf{x}^{(k)}, \;\; k = 1, 2, \cdots$$

1. extract the diagonal of A and denote it by N^J,
2. obtain P^J using $P^J = N^J - A$.

Example (Harder than the Jacobi matrices!)

Example

Write the Gauss Seidel method applied to the linear system (1)–(3) using the vector-matrix format (4):

$$N^{GS}\mathbf{x}^{(k+1)} = b + P^{GS}\mathbf{x}^{k}, \quad b = [10, 19, 0]^{T}.$$

Recall,

$$x_1{}^{(k+1)} = \frac{10 - x_2{}^{(k)} + x_3{}^{(k)}}{9}$$

$$x_2{}^{(k+1)} = \frac{19 - 2x_1{}^{(k+1)} - 3x_3{}^{(k)}}{10}$$

$$x_3{}^{(k+1)} = \frac{-(3x_1{}^{(k+1)} + 4x_2{}^{(k+1)})}{11}.$$

Solution

We start in the usual way of multiplying the first equation by 9, second equation by 10 and third by 11 to obtain

$$9{x_1}^{(k+1)} = 10 - {x_2}^{(k)} + {x_3}^{(k)}$$
$$10{x_2}^{(k+1)} = 19 - 2{x_1}^{(k+1)} - 3{x_3}^{(k)}$$
$$11{x_3}^{(k+1)} = -(3{x_1}^{(k+1)} + 4{x_2}^{(k+1)}).$$

Remember all terms with superscript $k+1$ belong to the LHS thus, rearranging gives us

$$9{x_1}^{(k+1)} = 10 - {x_2}^{(k)} + {x_3}^{(k)}$$
$$2{x_1}^{(k+1)} + 10{x_2}^{(k+1)} = 19 - 3{x_3}^{(k)}$$
$$3{x_1}^{(k+1)} - 4{x_2}^{(k+1)} 11{x_3}^{(k+1)} = 0.$$

We already have $b = [10, 19, 0]^T$. What is N^{GS} and P^{GS}?

$$
\begin{aligned}
9x_1^{(k+1)} &= 10 - x_2^{(k)} + x_3^{(k)} \\
2x_1^{(k+1)} + 10x_2^{(k+1)} &= 19 - 3x_3^{(k)} \\
3x_1^{(k+1)} - 4x_2^{(k+1)} 11x_3^{(k+1)} &= 0.
\end{aligned}
$$

It is easier to obtain P^{GS} in this case!

$$
P^{GS} = \begin{bmatrix} 0 & -1 & -1 \\ 0 & 0 & -3 \\ 0 & 0 & 0 \end{bmatrix},
$$

since $P^{GS}\mathbf{x}^{(k)} = \begin{bmatrix} -x_2^{(k)} + x_3^{(k)} \\ -3x_3^{(k)} \\ 0 \end{bmatrix}$ (verify!)

$$N^{GS} = P^{GS} + A = \begin{bmatrix} 0 & -1 & -1 \\ 0 & 0 & -3 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 9 & 1 & 1 \\ 2 & 10 & 3 \\ 3 & 4 & 11 \end{bmatrix} = \underbrace{\begin{bmatrix} 9 & 0 & 0 \\ 2 & 10 & 0 \\ 3 & 4 & 11 \end{bmatrix}}_{\text{Lower half of A!}}$$

<u>Observations</u>

- For the Gauss Seidel iterative method too, the matrices N^J and P^J stay unchanged!
- Notice the zero diagonal entries for P^{GS} too! Common feature with Jacobi matrices!
- The diagonal entries of N^{GS} and A are the same! Common feature with Jacobi matrices!

Convergence Analysis

Theorem (Convergence Condition)

For any iterative method

$$N\mathbf{x}^{(k+1)} = b + P\mathbf{x}^{(k)},$$

to solve $Ax = b$, the condition for convergence is

$$\|N^{-1}P\| < 1$$

for all choices of initial guess $\mathbf{x}^0$ and b!

Observations

- For Jacobi method, this condition is equivalent to requiring

$$\sum_{j=1, j\neq i}^{n} |a_{ij}| < |a_{ii}|, \quad i = 1, \cdots n.$$

A matrix $A = (a_{ij})_{i,j=1}^{n}$ satisfying the above condition is called diagonally dominant.