Math 4329

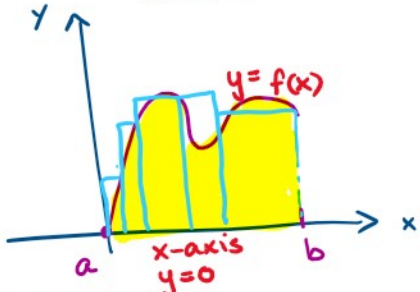
Thursday, April 16, 2020 4:38 PM

Topic → How to **numerically** approximate the area of a region?

Area = I

$$I = \int_a^b (f(x) - 0)\, dx$$

$$I = \int_a^b f(x)\, dx$$




Trapezoidal Rule or Simpson's Rule

$f(x) = 4 \quad a=0 \quad b=2$

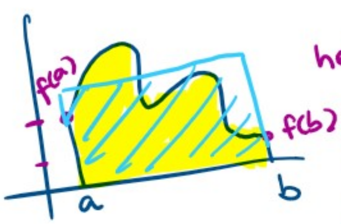

known formula: length * width

$4 * 2$

$I = 8$

$$I = \int_0^2 f(x)\, dx = 4\int_0^2 dx = 8$$

$$I = \int_a^b f(x)dx \longrightarrow \left(\frac{f(a)+f(b)}{2}\right)(b-a)$$

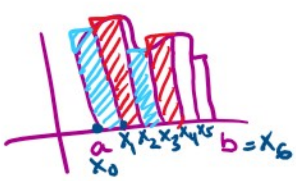



height $= \dfrac{f(a)+f(b)}{2}$

width $= b-a$

$$\left(\frac{f(a)+f(b)}{2}\right)(b-a)$$

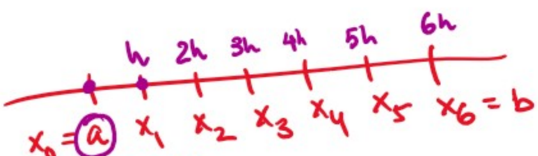
Trapezoidal Rule




$x_0\ x_1\ x_2 \ldots x_6 \qquad h = \dfrac{b-a}{5}$

$$I = \int_a^b f(x)\, dx = \frac{h}{2}\left(f(a) + 2f(x_1) + 2f(x_2) \ldots + f(x_6)\right)$$

= Trapezoidal Rule




$x_0 = a + 0h$

$x_0 = a \quad x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \quad x_6 = b$

$x_0 = a + 0h$

$x_1 = a + h = a + \frac{1}{6}(b-a)$

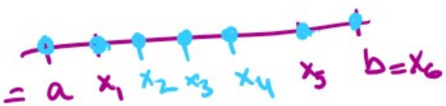
$x_2 = a + 2h$

$x_3 = a + 3h$

$x_4 = a + 4h$

$x_5 = a + 5h$

$x_6 = a + 6h$, $h = \frac{b-a}{6}$ gap bet. 2 consecutive eqidistant pts bet. a and b




$h = (b-a)/6$

$$T_6(f) = \frac{h}{2}\Big( f(\overset{x_0}{a}) + 2f(x_1) + 2f(x_2) + 2f(x_3) + 2f(x_4) + 2f(x_5) + f(\underset{x_6}{b}) \Big)$$

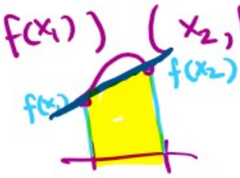
$$\int_a^b f(x)\,dx = \int_a^{x_1} f(x)\,dx + \int_{x_1}^{x_2} f(x)\,dx + \dots + \int_{x_5}^{x_6} f(x)\,dx$$

$$\approx \int_a^{x_1} p^{(0)}(x)\,dx + \int_{x_1}^{x_2} p^{(1)}(x)\,dx + \dots + \int_{x_5}^{x_6} p^{(5)}(x)\,dx$$

where $p^{(0)}(x)$ line joining $(a, f(a))$ $(x_1, f(x_1))$

$p^{(1)}(x)$ line/linear poly interpolating

$(x_1, f(x_1))$ $(x_2, F(x_2))$




$$\int_{x_1}^{x_2} f(x)\,dx \approx \int_{x_1}^{x_2} p^{(1)}(x) = \int f(x_2)\frac{(x - x_1)}{(x_2 - x_1)} + \int f(x_1)\frac{(x_2 - x)}{(x_2 - x_1)}$$

$x_2 - x_1 = h$

$$\int_{x_1}^{x_2} p^{(1)}(x)\,dx = \frac{f(x_2)}{h}\int_{x_1}^{x_2}(x - x_1) + \frac{f(x_1)}{h}\int_{x_1}^{x_2}(x_2 - x)\,dx$$

$$= \frac{f(x_2)}{h}\left(\frac{x^2}{2} - x_1 x\right)\Big|_{x_1}^{x_2} + \frac{f(x_1)}{h}\left(x_2 x - \frac{x^2}{2}\right)\Big|_{x_1}^{x_2}$$

$\vdots$

$$\int_{x_1}^{x_2} p^{(1)}(x)\,dx = h\left(\frac{f(x_1) + f(x_2)}{2}\right)$$

$f(x_0)$, $f(x_1)$

$$\int_{x_1} p^{(1)}(x)\,dx = h\left(\frac{f(x_1)+f(x_2)}{2}\right)$$


$$\int_{x_0}^{x_1} p^{(0)}(x)\,dx = h\left(\frac{f(x_0)+f(x_1)}{2}\right)$$

$$\int_{x_1}^{x_2} p^{(1)}(x)\,dx = h\left(\frac{f(x_1)+f(x_2)}{2}\right)$$

$\vdots$

$$\int_{x_5}^{x_6} p^{(5)}(x)\,dx = h\left(\frac{f(x_5)+f(x_6)}{2}\right)$$

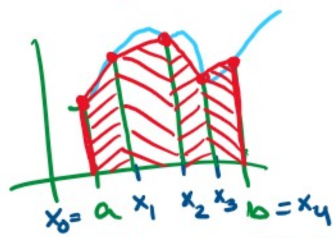
$$\int_{x_0}^{x_1} p^{(0)}(x) + \int_{x_1}^{x_2} p^{(1)}(x) + \cdots \int_{x_5}^{x_6} p^{(5)}(x)\,dx$$

$$= h\left(\frac{f(x_0)+f(x_1)}{2} + \frac{f(x_1)+f(x_2)}{2} + \cdots \frac{f(x_5)+f(x_6)}{2}\right)$$

$$= \frac{h}{2}\left(f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + f(x_6)\right)$$

$$\int_a^b f(x)\,dx$$

$$= \int_{x_0}^{x_1} f\,dx + \int_{x_1}^{x_2} f\,dx + \int_{x_2}^{x_3} f\,dx + \int_{x_3}^{x_4} f\,dx$$




$x_0 = a$

$x_1 = a+h$

$x_2 = a+2h$

$x_3 = a+3h$

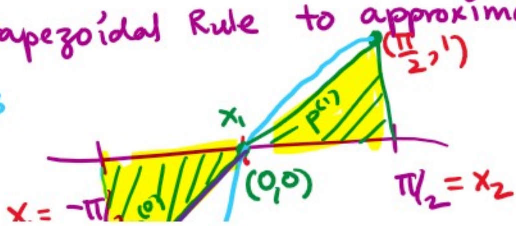
$x_4 = a+4h$

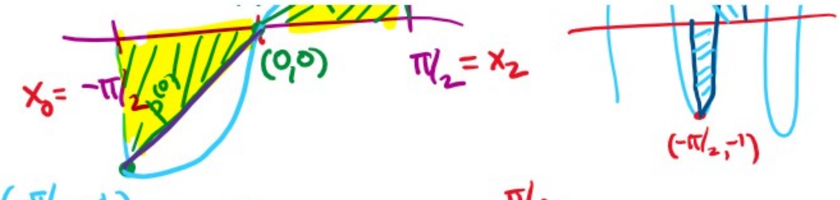
$h = (b-a)/4$

$$I = \int_{-\pi/2}^{\pi/2} \sin x\,dx = 0$$

Use Trapezoidal Rule to approximate $I$

using $n=3$




$$\int_{-\pi/2}^{\pi/2} \sin x \, dx = \int_{-\pi/2}^{0} \sin x \, dx + \int_{0}^{\pi/2} \sin x \, dx$$

$n \to$ # of subintervals

formula 3 pts $\to n = 2$ $\quad a = -\pi/2,\ b = \pi/2$

$$\boxed{h = \frac{\pi/2 - (-\pi/2)}{2} = \pi/2}$$

$$I \approx T_2(f) = \frac{h}{2}\left( f(x_0) + 2f(x_1) + f(x_2) \right)$$

$$T_2(f) = \frac{\pi/2}{2}\left( f(-\pi/2) + 2f(-\pi/2 + h) + f\left(-\frac{\pi}{2} + 2h\right)\right)$$

$$= \frac{\pi}{4}\left( \overset{-1}{\sin(-\pi/2)} + 2\overset{0}{\sin 0} + \overset{1}{\sin \pi/2} \right)$$

$$T_2(f) = \pi/4 \ (-1 + 1) = 0$$

Total # of points = 3

$\frac{\pi/3}{2}$

$x_0 = -\pi/2 \qquad 0 = x_1 \qquad \pi/2 = x_2$

$\sin(-x) = -\sin(x)$

total number of points = 5

$x_0 \quad -\pi/4 \quad 0 \quad \pi/4 \quad \pi/2 \quad x_4 = b$

$-\pi/2 \quad x_1 \quad x_2 \quad x_3 \quad \pi/2$

$x_0 = a$

$x_1 = a + h \ \ldots \quad x_4 = a + 4h \Rightarrow b = a + 4h \Rightarrow h = \frac{1}{4}(b - a)$

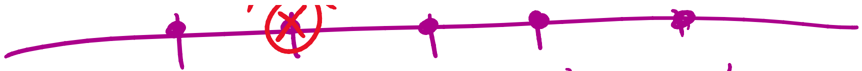
$h = ?$ $\quad h = \dfrac{\text{length of } [-\pi/2, \pi/2]}{\text{one less than \# of points} \to \text{\# of subintervals}}$

$$h = \pi/4$$

$x_0 = -\pi/2 = -2\pi/4$

$x_1 = -\pi/2 + h = -2\pi/4 + \pi/4 =$

$-\pi/ \quad -\pi/ \quad 0 \quad \pi/ \quad \pi/ = x_4$



$x_0 = -\pi/2 \quad -\pi/4 \quad 0 \quad \pi/4 \quad \pi/2 = x_4$

$n$ = # of subintervals (one less than total # of points)

$f(x) = \sin x$

$$T_4(f) = \frac{\pi/4}{2}\Big( f(-\pi/2) + 2\big(f(-\pi/4) + f(0) + f(\pi/4)\big) + f(\pi/2) \Big)$$

$$\cancel{\sin(-\pi/2)} + 2\cancel{f(-\pi/4)} + 2f(0) + 2\cancel{f(\pi/4)} + \cancel{\sin(\pi/2)}$$

$\cancel{-1} \quad -\cancel{f(\pi/4)} \quad f(\pi/4) \quad 1$

$2\sin 0 = 0$

$$\int_0^{\pi/2} \sin x \, dx$$

$x_1$

$-\pi/2 \quad x_0 = 0 \quad \pi/4 \quad \pi/2 = x_2$

$f(x) = \underline{\underline{\sin x}}$ $\qquad a = 0 \qquad b = \pi/2$

$$h = \frac{b-a}{2} = \pi/4$$

$$T_2(f) = \frac{\pi/4}{2}\Big( f(0) + 2f(\pi/4) + f(\pi/2) \Big)$$

$0 \qquad 2\sin \pi/4 \qquad + 1$

$2\sqrt{2}/2$

$$= \frac{\pi}{8}\left(0 + 2\sqrt{2}/2 + 1\right) = 0.94805$$

$$\int_0^{\pi/2} \sin x \, dx = -\cos x \Big|_0^{\pi/2} = -(0-1) = 1$$

$$\int_0^{2\pi} \sin x \, dx = -\cos x \Big|_0^{2\pi} = 0$$

$0 \qquad 2\pi = x_n$

$x_0 = 0$

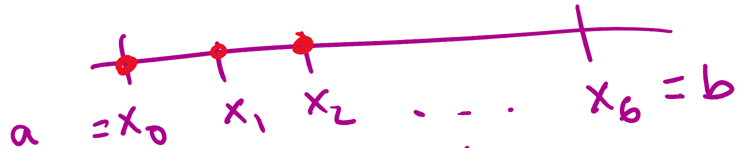
# of subintervals

$$T_n(f) = \frac{h}{2}\Big( f(x_0) + 2f(x_1) + \dots + 2f(x_{n-1}) + f(x_n) \Big)$$

# of subinter…

$$T_n(f) = \frac{h}{2}\left( f(x_0) + 2f(x_1) + \dots + 2f(x_{n-1}) + f(x_n) \right)$$

$$\int_a^b f(x)\,dx = \int_{x_0}^{x_1} f(x) + \dots \int_{x_5}^{x_6} f(x)\,dx$$




$$\int_a^b f(x)\,dx \approx \int_{x_0}^{x_1} p^{(0)}(x) + \int_{x_1}^{x_2} p^{(1)}(x) + \dots \int_{x_5}^{x_6} p^{(5)}(x)$$

Quadratic polynomials on $i = 0, \dots 5$