SPLINES.

Thursday, October 17, 2019 11:20 AM

#15

$$s(x) = \begin{cases} x^3 + 2x^2 + 1 = s_1 & 1 \le x \le 2 \\ 3x^2 + 4x \to s_1' & \\ -2x^3 + \beta x^2 - 36x + 25 \; s_2 & 2 \le x \le 3 \\ -6x^2 + 2\beta x - 36 = s_2'(x) & \end{cases}$$

unknown?

$s_2(2) = s_1(2)$

$-2 * 8 + 4\beta - 72 + 25 = 8 + 8 + 1$

$-16 - 72 + 25 + 4\beta = 17$

$4\beta = 80$

$\boxed{\beta = 20}$

$s_1'(2) \stackrel{?}{=} s_2'(2)$ and $s_1''(2) = s_2''(2)$

check: TRUE!

$s_1'(2) = 12 + 8 = 20$

$s_2'(2) = -24 + 4\beta - 36$

$= -60 + 80$

$= 20$

$s_1''(x) = 6x + 4$

$s_1''(2) = 18$

$s_2''(x) = -12x + 2\beta$

$= -24 +$

(16)

$$S(x) = \begin{cases} s_1 \to (x+1)^3 & -2 \le x \le -1 \\ s_2 \to ax^3 + bx^2 + cx + d & -1 < x < \boxed{1} \\ s_3 \to (x-1)^2 & \boxed{1} \le x \le 2. \end{cases}$$

4 unknowns Use spline conditions

s_1 at $x=-1$	s_2	at $x=1$ s_3
$(x+1)^3$ 0	$ax^3 + bx^2 + cx + d$	$(x-1)^2$ 0
$3(x+1)^2$ 0	$3ax^2 + 2bx + c$	$2(x-1)$ 0
$6(x+1)$ 0	$6ax + 2b$	$\boxed{2}$

Set up linear sys 4 eqns 4 unknowns:

$s_2(-1) = s_1(-1) \to -a + b - c + d = 0$

$s_2(1) = s_3(1) \to a + b + c + d = 0$

$s_2(1) = s_3(1) \longrightarrow a + b + c + d \ldots$

$s_2'(-1) = s_1'(-1) \longrightarrow 3a - 2b + c = 0$

$s_2'(1) = s_3'(1) \longrightarrow 3a + 2b + c = 0$

$s_2''(-1) = s_1''(-1) \longrightarrow -6a + 2b = 0$

$s_2''(1) = s_1''(1) \longrightarrow 6a + 2b$ -

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$$S(x) = \begin{cases} (x+1)^3 & S_1(x) & -2 \le x \le -1 \\ ax^3+bx^2+cx+d & S_2(x) & -1 < x < \boxed{1} \\ (x-1)^2 & S_3(x) & \boxed{1} \le x \le 2 \end{cases}$$

figure out a, b, c & d so that s(x) is a natural cubic spline.

$S_1(x)$ $\quad$ $S_2(x)$ $\quad$ $S_3(x)$

We want to find a, b, c, d such that:

$$\lim_{x \to -1^+} S_2(x) = S_1(-1)$$

$$\lim_{x \to 1} S_2(x) = S_3(1)$$

$$S_2'(-1) = S_1'(-1)$$

$$S_2'(1) = S_3'(1)$$

$$S_2''(-1) = S_1''(-1)$$

$$S_2''(1) = S_3''(1)$$

S_1 at $x=-1$ $\quad$ S_2 $\quad$ $\boxed{\text{at } x=1}$ $\quad$ S_3

$(x+1)^3$ $\quad$ ax^3+bx^2+cx+d $\quad$ $(x-1)^2$

$S_1(-1) = 0 \leftarrow -a+b-c+d$ $x=-1$

$x=1$ $a+b+c+d \rightarrow 0$

$3(x+1)^2$ $\quad$ $3ax^2+2bx+c$ $\quad$ $2(x-1)$

$S_1'(-1) = 0 \leftarrow 3a-2b+c$

$3a+2b+c \rightarrow 0$

$6(x+1)$ $\quad$ $6ax+2b$ $\quad$ 2

$S_1''(-1) = 0 \xleftarrow{\text{equality}} -6a+2b$

$6a+2b \rightarrow \boxed{2}$

Linear sys. of equations :

Linear sys. of equations :

$$\begin{aligned}
-a+b-c+d &= 0 \\
a+b+c+d &= 0 \\
3a-2b+c &= 0 \\
3a+2b+c &= 0
\end{aligned}$$

$$\boxed{\begin{aligned}
-6a+2b &= 0 \\
6a+2b &= 2
\end{aligned}}$$

Add

$$\begin{aligned}
-6a+2b &= 0 \\
6a+2b &= 2 \\
0+4b &= 2 \\
\boxed{b = 1/2}
\end{aligned}$$

use $-6a+2b=0$ and $b=1/2$

$$-6a+1 = 0$$

$$\boxed{a = 1/6}$$

Solve for c

$$\begin{aligned}
3a-2b+c &= 0 \\
3a+2b+c &= 0
\end{aligned}$$

using $a=1/6$ and $b=1/2$ in $3a-2b+c=0$

$$\begin{aligned}
3(1/6)-1+c &= 0 \\
1/2-1+c &= 0 \\
-1/2+c &= 0
\end{aligned}$$

$-1/2 \quad + \quad c \qquad\qquad = 0$

$$\boxed{c = 1/2}$$

Solve for d using $a = 1/6$, $b = 1/2$, $c = 1/2$

in $\quad -a + b - c + d = 0$

$$-1/6 + 1/2 - 1/2 + d = 0 \rightarrow \boxed{d = 1/6}$$

$$S(x) = \begin{cases} (x+1)^3 & -2 \le x \le -1 \\ x^3/6 + \frac{x^2}{2} + \frac{x}{2} + 1/6 & -1 < x < 1 \\ (x-1)^2 & 1 \le x \le 2. \end{cases}$$

Numerical Integration:

Answering a mathematical problem numerically!

evaluation of a "complicated function"

$$f(x) \approx P_N(x).$$

$\downarrow$

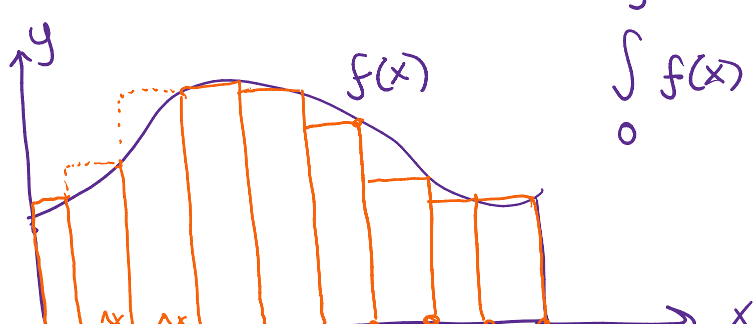
function involving only

$+, -, *, \div$

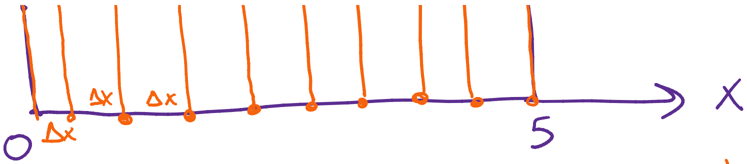
$$|f(x) - P_N(x)| \le ?$$

We want to $\displaystyle\int_a^b f(x)\,dx \approx \int_a^b P_N(x)\,dx$

Simpson's Rule, Trapezoidal Rule



$$\int_0^5 f(x)\,dx$$



Goal of chapter → Derive / Design integration formulas that approximate $\int_a^b f(x)\,dx$.

$$I = \int_0^1 e^{-x^2}\,dx$$

Apply Trapezoidal Rule $T_3(f)$ to approximate I.

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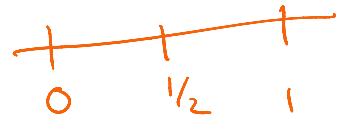
$$T_2(f) = \frac{h}{2}\left(f(a) + 2f\left(\frac{a+b}{2}\right) + f(b)\right)$$



$h = \frac{b-a}{2}$, $\boxed{f(x) = e^{-x^2}}$ $a = 0$, $b = 1$

$h = \frac{1-0}{2} = 0.5$

$$T_2(f) = \frac{0.5}{2}\left(f(0) + 2f(1/2) + f(1)\right)$$

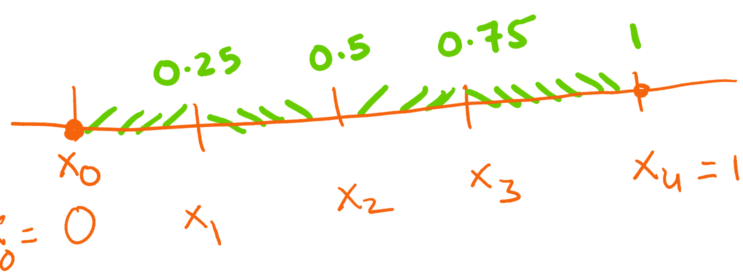
$$= 0.25\left(e^{-0^2} + 2 * e^{-0.5^2} + e^{-1}\right)$$

$$= 0.25\left(\frac{1}{e^{0^2}} + 2\frac{1}{e^{0.5^2}} + \frac{1}{e}\right)$$

$$e^{-x^2} = \frac{1}{e^{x^2}}$$

$$\boxed{T_2(f) = 0.73}$$

use $T_4(f)$ to approximate $\int_0^1 e^{-x^2}dx$



$n=4 \Rightarrow$ 4 subintervals $\rightarrow$ 5 points start 0 end 1

h = gap bet x_0 & x_1, x_1, x_2 .. $= \frac{b-a}{n} = 1/4$

$$T_4(f) = \frac{h}{2}\left(f(0) + 2f(0.25) + 2f(0.5) + 2f(0.75) + f(1)\right)$$

AW

$$= \frac{0.25}{2}\left(e^{-0^2} + 2e^{-0.25^2} + 2e^{-0.5^2} + 2e^{-0.75^2} + e^{-1^2}\right)$$

e^0

e^{-1}

$T_4(f) \approx 0.74298$

Homework 03 Math 4329

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1. $\{(-1,0), (0,2)\}$ is given data.

Find a & b satisfying $f(-1)=0$
$f(0)=2$.

$$\left.\begin{array}{l} a+be^{-1}=0 \\ a+be^{0}=2 \end{array}\right\} \Rightarrow \begin{array}{l} a+be^{-1}=0 \\ a+b=2 \end{array}$$ solve it.

2. Find $p(x)=ax^3+bx^2+cx+d$ by using

$$\left.\begin{array}{l} p(0)=y_1 \\ p(1)=y_2 \\ p'(0)=y_3 \\ p'(1)=y_4 \end{array}\right\} \Rightarrow \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

$p'(x)=3ax^2+2bx+c$

$p'(1)=3a+2b+c$

solve it for a,b,c,d gives answer

3.

	x_i	y_i
0	0	0
1	0.5	0.25
2	1	1
3	2	-1
4	3	-1

For $x_0=0$ $y_0=0$
$x_1=0.5$ $y_1=0.25$,

$$\boxed{P_1(x)=0.25\,\frac{(x-0)}{0.5}}$$

For $x_1=0.5$ $y_1=0.25$, $x_2=1$ $y_2=1$ $\Rightarrow$ $\boxed{P_2(x)=0.25\,\frac{(1-x)}{0.5}+\frac{x-0.5}{0.5}}$

For $x_2=1$ $y_2=1$, $x_3=2$ $y_3=-1$ $\Rightarrow$ $\boxed{P_3(x)=(2-x)+(-1)(x-1)}$

For $x_3=2$ $y_3=-1$, $x_4=3$ $y_4=-1$ $\Rightarrow$ $\boxed{P_4(x)=-(3-x)-(x-2)}$

3(b) Use the equations: $M_0=M_4=0$

$$\frac{(x_j-x_{j-1})}{}M_{j-1}+\frac{(x_{j+1}-x_{j-1})}{}M_j+\frac{(x_{j+1}-x_j)}{}M_{j+1}=\frac{y_{j+1}-y_j}{x_{j+1}-x_j}-\frac{y_j-y_{j-1}}{x_j-x_{j-1}}$$

5(b)

$$\left(\frac{x_j - x_{j-1}}{6}\right) M_{j-1} + \frac{(x_{j+1} - x_{j-1})}{3} M_j + \frac{(x_{j+1} - x_j)}{6} M_{j+1} = \frac{y_{j+1} - y_j}{x_{j+1} - x_j} - \frac{y_j - y_{j-1}}{x_j - x_{j-1}}$$

$j=1$ gives

$$\frac{\overset{0.5-0}{x_1 - x_0}}{6} M_0 + \frac{(\overset{1-0}{x_2 - x_0})}{3} M_1 + \frac{(\overset{1-0.5}{x_2 - x_1})}{6} M_2 = \frac{\overset{1-0.25}{y_2 - y_1}}{\underset{1-0.5}{x_2 - x_1}} - \frac{\overset{0.25-0}{y_1 - y_0}}{\underset{0.5-0}{x_1 - x_0}}$$

$j=2$ gives

$$\frac{\overset{1 \quad 0.5}{x_2 - x_1}}{6} M_1 + \frac{(\overset{2 \quad 0.5}{x_3 - x_1})}{3} M_2 + \frac{(\overset{2 \quad 1}{x_3 - x_2})}{6} M_3 = \frac{\overset{-1-1}{y_3 - y_2}}{\underset{2-1}{x_3 - x_2}} - \frac{\overset{1-0.25}{y_2 - y_1}}{\underset{1-0.5}{x_2 - x_1}}$$

$j=3$ gives

$$\frac{\overset{2 \quad 1}{x_3 - x_2}}{6} M_2 + \frac{(\overset{3 \quad 1}{x_4 - x_2})}{3} M_3 + \frac{(\overset{3 \quad 2}{x_4 - x_3})}{6} M_4 = \frac{\overset{-1+1}{y_4 - y_3}}{\underset{3-2}{x_4 - x_3}} - \frac{\overset{-1-1}{y_3 - y_2}}{\underset{2-1}{x_3 - x_2}}$$

$$\tfrac{1}{3} M_1 + \frac{0.5}{6} M_2 = \frac{\overset{1.5}{\cancel{0.75}}}{\cancel{0.5}} - \frac{\cancel{0.25}\,1}{\cancel{0.5}\,2} = 1 - 0.5 = 0.5$$

$$\frac{0.5}{6} M_1 + \frac{1.5}{3} M_2 + \frac{M_3}{6} = -2 - \frac{\cancel{0.75}\,1.5}{\cancel{0.5}} = -3.5$$

$$\tfrac{1}{6} M_2 + \frac{2}{3} M_3 = 0 - (-2) = 2$$

Splines
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Construction of splines, checking if a function is a spline.

#16

$$S(x) = \begin{cases} (x+1)^3 & s_1 & -2 \le x \le -1 \\ ax^3+bx^2+cx+d & s_2 & -1 < x < 1 \\ (x-1)^2 & s_3 & 1 \le x \le 2 \end{cases}$$

choose a, b, c & d so that S(x) is a spline.

Set up linear system in unknowns a, b, c & d using spline conditions.

$-a+b-c+d = S_2(-1) = S_1(-1) = 0$

$a+b+c+d = S_2(1) = S_1(1) = 0$

$S_2'(x)$: $\boxed{3ax^2+2bx+c}$

$S_2'(-1) = S_1'(-1)$

$\boxed{3a-2b+c = 0}$

$S_2'(1) = S_3'(1)$

$\boxed{3a+2b+c = 0}$

$S_1(x) = (x+1)^3$

$\boxed{S_1'(x) = 3(x+1)^2}$

$S_3(x) = (x-1)^2$

$S_3'(x) = 2(x-1)$

$S_1''(x) = 6(x+1)$

$S_3''(x) = 2$

$\boxed{6ax+2b}$

$-6a+2b \to S_2''(-1) = S_1''(-1) = 0$

$\boxed{6a+2b} \to S_2''(1) = S_3''(1) = 2$

We arrive at the foll. sys:

$$\begin{aligned} -a+b-c+d &= 0 \\ a+b+c+d &= 0 \\ 3a-2b+c &= 0 \\ 3a+2b+c &= 0 \\ -6a+2b &= 0 \\ 6a+2b &= 2 \end{aligned}$$

add them gives

$0+4b = 2$

or $\boxed{b = 1/2}$

Use $-6a+2b = 0$ & $b = 1/2$

$-6a+1 = 0$ or $a = 1/6$

$$-6a+1 = 0 \quad \text{or} \quad a = 1/6$$

Plug in $a = 1/6, b = 1/2$ in $3a + 2b + c = 0$

$$1/2 + 1 + c = 0 \Rightarrow \boxed{c = -3/2}$$

finally plug in $a = 1/6$, $b = 1/2$ $c = -3/2$ in

$$a + b + c + d = 0$$

to give d:

$$1/6 + 1/2 - 3/2 + d = 0$$

$$1/6 - 1 + d = 0$$

$$\boxed{d = 5/6}$$

Answer: $S(x) = \begin{cases} (x+1)^3 & -2 \le x \le -1 \\ x^3/6 + \frac{x^2}{2} - \frac{3}{2}x + \frac{5}{6} & -1 < x < 1 \\ (x-1)^2 & 1 \le x \le 2. \end{cases}$

Chap 5

Numerical Integration & Differentiation

Numerically answering a math. problem

$$\int_0^1 e^{-x^2}\,dx \approx \int_0^1 p_N(x)\,dx$$

choose $x_0, x_1, \ldots x_{10}$ points & interpolate e^{-x^2} at these chosen points. These points must lie bet. 0 & 1.

Construct $P_{10}(x)$ interpolating e^{-x^2}.

$$\int_0^1 e^{-x^2}\,dx \approx \int_0^1 p_{10}(x)\,dx$$

error in this integration is related to error between e^{-x^2} & $p_{10}(x)$.

error in this integration is related to error between e^{-x^2} & $p_{10}(x)$.

Another way: $e^{-x^2} = P_n(x) + R_n(x)$

where $P_n(x)$ is Taylor poly & $R_n(x) = \frac{(x-0)^{n+1}}{(n+1)!} f^{(n+1)}(c)$.

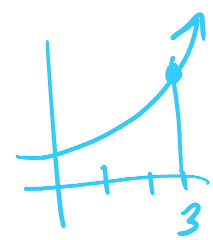
$$\int_0^1 e^{-x^2}\,dx = \int_0^1 P_n(x)\,dx + \int_0^1 R_n(x)\,dx$$

as n becomes large,

$\left| \int_0^1 e^{-x^2}\,dx - \int_0^1 P_n(x)\,dx \right|$ becomes much smaller.

$$E = \int_0^1 \frac{x^8}{4!} e^c \, dx \longrightarrow E = \int_2^3 \frac{x^8}{4!} e^c \, dx$$

$c = 3$



$<$

$$E = \int_0^1 \frac{x^8}{4!} e^{-c} \, dx \longrightarrow c = 0 \qquad \int_{-1}^{1} \frac{x^8}{4!} e^{-c}$$

$$\boxed{T_n(f) = \frac{h}{2}\left(f(x_0) + 2f(x_1) \ldots + 2f(x_{n-1}) + f(x_n) \right)}$$

$a = x_0$, x_1, x_2, ..., x_{n-1}, $x_n = b$

$h = \frac{b-a}{n}$ → you need to figure this out!

Approximate $\int_0^1 e^{-x^2}\,dx$ using $T_4(f)$. → $n = 4$

Approximate $\int_0^1 e^{-x^2}\,dx$ using $T_4(f)$. $\rightarrow n=4$

$a=0 \qquad b=1 \qquad h=\frac{1-0}{4}=0.25$

$0=x_0 \quad x_1 \quad x_2 \quad x_3 \quad x_4=1$

$0 \quad 0.25 \quad 0.5 \quad 0.75 \quad 1$

$x_0=0$

$x_1=0+h$

$f(x)=e^{-x^2}=1/e^{x^2}$

$$T_4(f)=\frac{h}{2}\left[\underset{1}{e^{-0^2}}+2e^{-(0.25^2)}+2e^{-(0.5^2)}+2e^{-(0.75^2)}+\underset{e^{-1}}{e^{-1^2}}\right]$$

$$=\frac{0.25}{2}*\left[1+2e^{-(0.25)^2}+2e^{-0.25}+2e^{-(0.75)^2}+\frac{1}{e}\right]$$

$$\approx 0.7$$

Simpson Rule:

$$\int_a^b f(x)\,dx \approx \int_a^b p_2(x)\,dx$$

$f(x)\approx p_2(x)$

easily verify this by integrating Lag. poly

$a \quad \frac{a+b}{2} \quad b$

$$P_2(x)=f(a)L_0(x)+f\left(\frac{a+b}{2}\right)L_1(x)+f(b)L_2(x)$$

$$\int_a^b p_2(x)\,dx = \text{Simpson Rule.}$$

$\downarrow S_2(f)$

$$\int_a^b f(x)\,dx \approx \frac{h}{3}\left(f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right)$$

$a=x_0 \qquad \frac{a+b}{2}=x_1 \qquad x_2=b$

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SPLINES.

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#15

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unknown?

$S_2(2) = S_1(2)$

$-2*8 + 4\beta - 72 + 25 = 8+8+1$

$-16-72+25+4\beta = 17$

$4\beta = 80$

$\boxed{\beta = 20}$

$S_1'(2) \overset{?}{=} S_2'(2)$ and $S_1''(2) = S_2''(2)$

check: TRUE!

$S_1'(2) = 12+8 = 20$

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$= -60 + 80$

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(16)

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4 unknowns use spline conditions

S_1 at $x=-1$	S_2	S_3 at $x=1$
$(x+1)^3$ 0	$ax^3 + bx^2 + cx + d$	$(x-1)^2$ 0
$3(x+1)^2$ 0	$3ax^2 + 2bx + c$	$2(x-1)$ 0
$6(x+1)$ 0	$6ax + 2b$	$\boxed{2}$

Set up linear sys 4 eq^ns 4 unknowns:

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$S_2(1) = S_3(1) \to a + b + c + d = 0$

$S_2'(-1) = S_1'(-1) \to 3a - 2b + c = 0$

$S_2'(1) = S_3'(1) \to 3a + 2b + c = 0$

$S_2'(-1) = S_1'(-1) \longrightarrow 3a - 2b + c$

$S_2'(1) = S_3'(1) \longrightarrow 3a + 2b + c \qquad = 0$

$S_2''(-1) = S_1''(-1) \longrightarrow -6a + 2b \qquad = 0$

$S_2''(1) = S_1''(1) \longrightarrow 6a + 2b \qquad -$