Int. Formula

Tuesday, October 22, 2019 1:39 PM

$$I = \int_0^2 \frac{dx}{1+x^2}$$

$f(x) = \dfrac{1}{1+x^2}$ $\quad a = 0 \quad b = 2$

Approximate I by $T_4(f)$ and $S_4(f)$.

$\boxed{n=4}$

$$\begin{array}{ccccc} x_0 = 0 & x_1 & x_2 & x_3 & 2 = b \\ & 0.5 & 1 & 1.5 & \parallel \\ & & & & x_4 \end{array}$$

$$h = \frac{b-a}{4} = \frac{2-0}{4} = 1/2$$

$$T_4(f) = \frac{h}{2}\left(f(a) + 2f(x_1) \dots + 2f(x_3) + f(b)\right]$$

$$= \frac{0.5}{2}\left(\frac{1}{1+0^2} + \frac{2}{1+0.5^2} + \frac{2}{1+1^2} + \frac{2}{1+1.5^2} + \frac{1}{1+2^2}\right]$$

Simplify & obtain approximation!

$$\begin{array}{ccccc} x_0 & x_1 & x_2 & x_3 & x_4 \\ a = 0 & 0.5 & 1 & 1.5 & 2 = b \end{array}$$

$$S_4(f) = \frac{h}{3}\left(f(a) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(b)\right]$$

$$= \frac{0.5}{3}\left[\frac{1}{1+0^2} + \frac{4}{1+0.5^2} + \frac{2}{1+1^2} + \frac{4}{1+1.5^2} + \frac{1}{1+2^2}\right]$$

Simplify & obtain the approxm to I!

How large should n be so that

$$|I - T_n(f)| < 5 \times 10^{-6}$$

$$\left| I - T_n(f) \right| < 5 \times 10^{-6}$$

where $I = \int_0^2 \frac{1}{1+x^2}\,dx$ and $f(x) = \frac{1}{1+x^2}$?

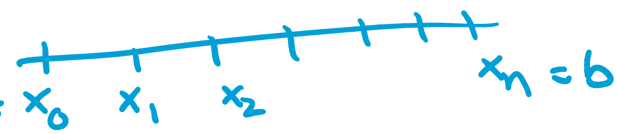
<u>Answer:</u> <u>Recall:</u>

$$I - T_n(f) = -h\frac{(b-a)}{12} f''(c_n)$$

c_n is unknown bet. a & b.

without calculating $T_n(f)$ we want to know how large n should be!

Recall $h = \frac{b-a}{n}$



$\int_a^b f(x)\,dx$

$$I - T_n(f) = -\frac{(2-0)}{12} h^2 \boxed{f''(c_n)}$$

?

$$f(x) = \frac{1}{1+x^2} \qquad f'(x) = -\frac{1}{(1+x^2)^2}(2x)$$

$$f''(x) = \frac{-2*(1+x^2)^2 - 2(1+x^2)(2x)(-2x)}{(1+x^2)^4}$$

$$= \frac{-2\cancel{(1+x^2)}\left((1+x^2) - 2x*2x\right)}{\ldots}$$

$$= \frac{-2\cancel{(1+x^2)}\left((1+x^2) - 2x * 2x\right)}{(1+x^2)^{\cancel{4}\,3}}$$

$$f''(x) = \frac{-2}{(1+x^2)^3}\left(1-3x^2\right) = \frac{6x^2-2}{(1+x^2)^3}$$

$f''(c_n)$

Back to formula:

$$\left|I - T_n(f)\right| = \left| -\frac{2}{12}h^2 \frac{(6c_n^2-2)}{(1+c_n^2)^3}\right| \quad 0 \le c_n \le 2.$$

$$= \frac{h^2}{6} \frac{|6c_n^2-2|}{(1+c_n^2)^3} \quad 0 \le c_n \le 2$$

$c_n = 0$ gives max. for $\dfrac{|6c_n^2-2|}{(1+c_n^2)^3}$

$$= \frac{h^2}{6} \; 2 = \frac{h^2}{3}$$

find n : $\quad \dfrac{h^2}{3} < 5 \times 10^{-6}$

$$\frac{\left(\frac{2-0}{n}\right)^2}{3} < 5 \times 10^{-6}$$

looking for $\quad \dfrac{4}{3n^2} < 5 \times 10^{-6}$

by some calculations

$< n^2$

by some calculations

$$\frac{4}{3(5\times10^{-6})} < n^2$$

after some simplification $\Rightarrow$ $n \geq 516.4$

$$\boxed{n \geq 517}$$

How large should n be so that

$$|I - S_n(f)| < 5\times10^{-6} \;?$$

formula: $I - S_n(f) = -\frac{(b-a)}{180} h^4 \underbrace{f^{(IV)}(c_n)}_{\text{fourth order derivative!}}$

See the notes!

Other Num. Integration formulas

$$I = \int_a^b f(x)\,dx$$

$a = x_0$ $\quad$ $x_n = b$

$T_n(f) \rightarrow w_0 f(x_0) + w_1 f(x_1) + \dots + w_n f(x_n)$

$S_n(f) \nearrow$

example: $T_n(f) = \underset{w_0}{\frac{h}{2}} f(x_0) + \underset{w_1}{h} f(x_1) + \dots + \underset{w_{n-1}}{h} f(x_{n-1}) + \underset{w_n}{\frac{h}{2}} f(x_n)$

$$\int_{-1}^{1} f(x)\,dx \approx \boxed{\hat{I}(f)} \rightarrow \text{some integration formula.}$$

$\int_{-1}^{1} f(x)\,dx \approx \boxed{I(f)}$, some integration formula.

"Gaussian" Integration Rule is characterized by $\int_{-1}^{1} f(x)dx$

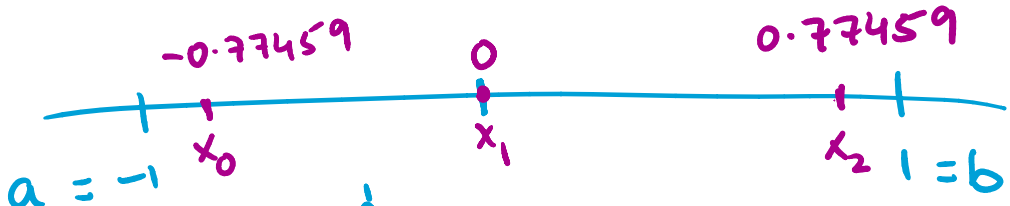
Quadrature Rule

Table 5.7, Text book

$\tilde{I}_n(f) = w_0 f(x_0) + w_1 f(x_1) \dots + w_{n-1} f(x_{n-1})$

n=2

n	x_i	w_i
2	$x_0 = -0.5773$	$w_0 = 1$
	$x_1 = 0.5773$	$w_1 = 1$
3	$x_0 = -0.77459$	$w_0 = 0.5555$
	$x_1 = 0$	$w_1 = 0.8888$
	$x_2 = 0.77459$	$w_2 = 0.5555$



$\int_{-1}^{1} f(x)\,dx$

3 point Gaussian Quad. Rule/formula.

$\tilde{I}_3(f) = 0.5555\, f(-0.77459) + 0.8888\, f(0) + 0.5555\, f(0.77459)$

example $\int_{-1}^{1} \frac{dx}{1+x^2}$ using 3 point Gaussian Quad formula,

$$\tilde{I}_3(f) = \frac{0.5555}{1+(-0.77459)^2} + \frac{0.8888}{1+0^2} +$$

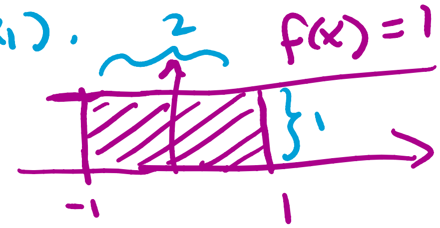
$$\frac{0.5555}{1+(0.77459)^2}$$

simplify to get answer!

Idea behind Gaussian Quad. formula:

2-point Gaussian Quad. formula:

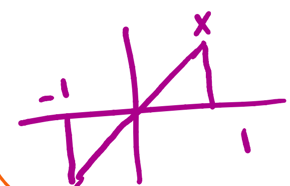
$$I = \int_{-1}^{1} f(x)\,dx \approx \underbrace{w_0 f(x_0) + w_1 f(x_1)}_{\tilde{I}_2(f)}.$$



We demand

for $f(x)=1$ $\quad \tilde{I}_2(f) = I = \int_{-1}^{1} 1\,dx = 2$ = area of rect.

$f(x)=x$ $\quad \tilde{I}_2(f) = \int_{-1}^{1} x\,dx = 0$



Similarly for

$f(x)=x^2$ & $f(x)=x^3$.

$w_0 f(x_0) + w_1 f(x_1) = 2$ $\quad f(x)=1$

$$\boxed{w_0 + w_1 = 2}$$

$w_0 f(x_0) + w_1 f(x_1) = 0$ $\quad f(x)=x$

$$\boxed{w_0 x_0 + w_1 x_1 = 0}$$

and derive two more equations using $f(x)=x^2$ & $f(x)=x^3$.

Solve the system gives:

$w_0 = w_1 = 1$

$x_0 = -0.5773$

$x_1 = 0.5773$

$f(x) = x^3$.

Solve the system gives: $w_0 = w_1 = 1$

$$\boxed{\begin{aligned} w_0 &= w_1 = 1 \\ x_0 &= -0.5773 \\ x_1 &= 0.5773 \end{aligned}}$$

This is exactly 2-point Gaussian Quad Rule.

Consider the following integration formula:

$$\tilde{I}(f) = c_0 f(0) + c_1 f(1)$$

which approximates

$$\int_0^1 f(x)\,dx \quad \text{for any choice of } f(x).$$

Determine c_0 and c_1 so that the formula $\tilde{I}(f)$ is $\boxed{\text{exact}}$ for $f(x)=1$ and $f(x)=x$.

$\Downarrow$

$f(x)=1$ in $\tilde{I}(f) = \int_0^1 1\,dx$

$f(x)=x$ in $\tilde{I}(f) = \int_0^1 x\,dx$

$f(x)=1$

$$c_0 f(0) + c_1 f(1) = \int_0^1 1\,dx = x\Big|_{x=0}^{1} = 1$$

$$c_0 * 1 + c_1 * 1 = 1$$

$$\boxed{c_0 + c_1 = 1}$$

$$f(x)=x \quad c_0 f(0) + c_1 f(1) = \int_0^1 x\,dx = \frac{x^2}{2}\Big|_{x=0}^{1}$$

$\hookrightarrow f(x) = x \quad c_0 f(0) + c_1 f(1) = \int_0 x\,dx = \frac{x^2}{2}\Big|_{x=0}$

$$\boxed{c_0 * 0 + c_1 = 1/2}$$

2 equations:

$$c_0 + c_1 = 1$$
$$c_1 = 1/2$$

gives: $c_0 = 1/2 = c_1$.

So integration formula that is exact for $f(x) = 1$ and $f(x) = x$ is

$$\hat{I}(f) = \underset{c_0}{1/2}\, f(0) + \underset{c_1}{1/2}\, f(1)$$

check:

$f(x) = 2$ and apply $\hat{I}(f)$

$$\hat{I}(f) = 1/2\, f(0) + 1/2\, f(1)$$
$$= 1/2 * 2 + 1/2 * 2$$
$$= 2$$

$$I(f) = \int_0^1 f(x)\,dx = \int_0^1 2\,dx = 2$$

So $I(f) = \hat{I}(f)$.

check $f(x) = x$ checking if $\hat{I}(f) = I(f)$

$\hat{I}(f) = 1/2\, f(0) + 1/2\, f(1) = 1/2$

Check:

$$\tilde{I}(f) = 1/2\, f(0) + 1/2\, f(1) = 1/2$$

$$I(f) = \int_0^1 x\,dx = \frac{x^2}{2}\Big|_{x=0}^{1} = 1/2$$

$$\boxed{I(f) = \tilde{I}(f)}$$

Note: for $p_1(x) = 4x + 3$

$$\int_0^1 p_1(x)\,dx = \int_0^1 (4x+3)\,dx = \cancel{4}^{2}\frac{x^2}{\cancel{2}} + 3x \Big|_{x=0}^{1}$$

$$= 5$$

$\tilde{I}(f) \longrightarrow f(x) = 4x+3$

$$= \frac{1}{2} f(0) + \frac{1}{2} f(1)$$

$$= \frac{1}{2}(4*0+3) + \frac{1}{2}(4+3) = \frac{3+7}{2} = 5$$

If $\hat{I}(f)$ is exact for $f(x) = 1$ and $f(x) = x$ then it is exact for any polynomial of degree 1.