

The 2-variable Second Derivative Test

$$\begin{aligned} z &= f(x, y) \\ x &= x_0 + \alpha s \\ y &= y_0 + \beta s \end{aligned}$$

By the chain rule, the first derivative in the direction $\langle \alpha, \beta \rangle$, where $\alpha^2 + \beta^2 = 1$, is

$$\frac{df}{ds} = \frac{\partial f}{\partial x} \frac{dx}{ds} + \frac{\partial f}{\partial y} \frac{dy}{ds} = \alpha f_x + \beta f_y$$

If $f_x = 0$ and $f_y = 0$ at (x_0, y_0) , then $\frac{df}{ds} = 0$ in any direction, and (x_0, y_0) is a critical point. To decide if it is a maximum, minimum, or neither, we need the **second** directional derivative:

$$\begin{aligned} \frac{d^2 f}{ds^2} &= \alpha \frac{df_x}{ds} + \beta \frac{df_y}{ds} = \alpha \left[\frac{\partial f_x}{\partial x} \frac{dx}{ds} + \frac{\partial f_x}{\partial y} \frac{dy}{ds} \right] + \beta \left[\frac{\partial f_y}{\partial x} \frac{dx}{ds} + \frac{\partial f_y}{\partial y} \frac{dy}{ds} \right] = \\ &= \alpha [\alpha f_{xx} + \beta f_{xy}] + \beta [\alpha f_{yx} + \beta f_{yy}] = \alpha^2 f_{xx} + 2\alpha\beta f_{xy} + \beta^2 f_{yy} = \\ &= \alpha^2 [f_{xx} + 2m f_{xy} + m^2 f_{yy}] \end{aligned}$$

where $m \equiv \frac{\beta}{\alpha}$. Note that we have just applied the chain rule to calculate the derivatives of the functions $f_x(x, y)$ and $f_y(x, y)$ with respect to s .

Now if

1. $f_{xx}f_{yy} - f_{xy}^2 < 0$, the discriminant $(2f_{xy})^2 - 4f_{xx}f_{yy}$ is positive, and the quadratic equation $f_{xx} + 2mf_{xy} + m^2f_{yy} = 0$ has two real roots. This means this quadratic factor, and thus $\frac{d^2 f}{ds^2}$, is negative for some m , and positive for other m , which means the surface is concave down in some directions, and concave up in others, and so the critical point (x_0, y_0) is a maximum in some directions, and a minimum in others, that is, it is a saddle point.
2. $f_{xx}f_{yy} - f_{xy}^2 > 0$, the discriminant is negative, and the quadratic equation has no real roots. Then either this quadratic factor, and thus $\frac{d^2 f}{ds^2}$, is positive for all m , or negative for all m .
 - a. If $f_{xx} > 0$, it is positive for $m = 0$ and thus it must be positive for all m , which means means the surface is concave up in all directions, and the critical point is a minimum.
 - b. If $f_{xx} < 0$, it is negative for $m = 0$ and thus negative for all m , which means the surface is concave down in all directions, and the critical point is a maximum.