# Interpolation

Lagrange polynomial interpolant to $(x_0, f(x_0)), (x_1, f(x_1)), (x_2, f(x_2)), (x_3, f(x_3))$:

$$\begin{aligned} L_3(x) &= \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} f(x_0) + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} f(x_1) \\ &+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} f(x_2) + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} f(x_3) \end{aligned}$$

Lagrange polynomial error bound:

$f(x) - L_3(x) = \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)}{4!} f^{iv}(\psi)$

Proof of error bound:

$G(t) \equiv f(t) - L_3(t) - \frac{(t-x_0)(t-x_1)(t-x_2)(t-x_3)}{(x-x_0)(x-x_1)(x-x_2)(x-x_3)} [f(x) - L_3(x)]$

$G = 0$ at $x_0, x_1, x_2, x_3, x$, so $G' = 0$ at at least 4 points, $G'' = 0$ at 3 points, $G''' = 0$ at 2 points, $G^{iv}(\psi) = 0$, where $min(x_0, x_1, x_2, x_3, x) \leq \psi \leq max(x_0, x_1, x_2, x_3, x)$.

$G^{iv}(t) = f^{iv}(t) - 0 - \frac{4!}{(x-x_0)(x-x_1)(x-x_2)(x-x_3)} [f(x) - L_3(x)]$

$G^{iv}(\psi) = f^{iv}(\psi) - \frac{4!}{(x-x_0)(x-x_1)(x-x_2)(x-x_3)} [f(x) - L_3(x)] = 0$

so

$f(x) - L_3(x) = \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)}{4!} f^{iv}(\psi) \quad \blacksquare$

For Taylor or Lagrange polynomial of degree $n$, if all interpolation points in an interval of width $h$, $|p_n(x) - f(x)| \leq \frac{1}{(n+1)!} max|f^{n+1}(x)|h^{n+1}$.

Safer to fix $n$ ($n = 3$?), let $h \rightarrow 0$.

$T_3$ matches $f(x_i), f'(x_i), f''(x_i), f'''(x_i)$; $T_3$ not continuous.
$L_3$ matches $f(x_i), f(x_i + h/3), f(x_i + 2h/3), f(x_i + h)$; $L_3$ continuous.
$H_3$ matches $f(x_i), f'(x_i), f(x_i + h), f'(x_i + h)$; $H_3, H_3'$ continuous.
$S_3$ matches $f(x_i), f(x_i + h)$; $S_3, S_3', S_3''$ continuous.

Find cubic spline interpolant $S_3(x_i) = f(x_i), i = 0, ..., N$.

Number of unknowns, $4 * N$.
Number of interpolation conditions, $2 * N$.
Number of continuity conditions (for $S_3', S_3''$), $2 * (N - 1)$.
Need two more "end" conditions, for example:

1. $S_3'(x_0) = f'(x_0)$ and $S_3'(x_N) = f'(x_N)$.
2. $S_3''(x_0) = f''(x_0)$ and $S_3''(x_N) = f''(x_N)$.
3. $S_3'''(x)$ continuous at $x_1$ and $x_{N-1}$ ("not-a-knot" condition).
4. $S_3''(x_0) = 0$ and $S_3''(x_N) = 0$ ("natural" end condition).

First 3 result in $O(h^4)$ accuracy, natural cubic spline only $O(h^2)$.