Quadrature

$\int_a^b f(x)\ dx = \sum_{i=1}^N \int_{x_{i-1}}^{x_i} f(x)\ dx$

(where $x_i = a + i\ h, \quad h = (b-a)/N$)

Simpson's local rule:

$\int_0^h f(x)\,dx \approx \int_0^h \left[\frac{(x-h/2)(x-h)}{(0-h/2)(0-h)} f(0) + \frac{(x-0)(x-h)}{(h/2-0)(h/2-h)} f(h/2) + \frac{(x-0)(x-h/2)}{(h-0)(h-h/2)} f(h)\right]\ dx$

$= \frac{h}{6} f(0) + \frac{4h}{6} f(h/2) + \frac{h}{6} f(h)$

Simpson's composite rule:

$\int_a^b f(x)\ dx \approx \sum_{i=1}^N \left[\frac{h}{6} f(x_{i-1}+0) + \frac{4h}{6} f(x_{i-1}+h/2) + \frac{h}{6} f(x_{i-1}+h)\right]$

Quadrature rule with sample points $r_1, r_2, r_3, ...$ and weights $w_1, w_2, w_3, ...$:

$\int_a^b f(x)\,dx \approx \sum_{i=1}^N [w_1 h f(x_{i-1}+r_1 h) + w_2 h f(x_{i-1}+r_2 h) + w_3 h f(x_{i-1}+r_3 h) + ...]$

Newton-Cotes formulas:

Name	r_1	r_2	r_3	r_4	r_5	w_1	w_2	w_3	w_4	w_5
Trapezoid	0	1				$\frac{1}{2}$	$\frac{1}{2}$			
Simpson	0	$\frac{1}{2}$	1			$\frac{1}{6}$	$\frac{4}{6}$	$\frac{1}{6}$		
—	0	$\frac{1}{3}$	$\frac{2}{3}$	1		$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$	
Boole	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1	$\frac{7}{90}$	$\frac{32}{90}$	$\frac{12}{90}$	$\frac{32}{90}$	$\frac{7}{90}$

Theorem: If a quadrature rule is exact for $f(x) = 1, x, x^2, ..., x^m$, then the global error is bounded by $C_m\ max_{(a,b)}|f^{(m+1)}(x)|h^{m+1}$

Proof: see chapter 5 video

MATLAB function implementing Simpson's rule

```
function int = simpson(f,a,b,n)
h = (b-a)/n;
w1 = 1./6.;
w2 = 4./6.;
w3 = 1./6.;
r1 = 0;
r2 = 0.5;
r3 = 1;
int = 0.0;
for i=1:n
   xim1 = a + (i-1)*h;
   int = int + w1*h*f(xim1+r1*h) ...
             + w2*h*f(xim1+r2*h) ...
             + w3*h*f(xim1+r3*h);
end
```

Experimental Order of Convergence

$E(h) = I(h) - I = M\ h^{\alpha}$

If exact integral I is known:

$ratio_1 \equiv \frac{E(h)}{E(h/2)} = \frac{M\ h^{\alpha}}{M\ (h/2)^{\alpha}} = 2^{\alpha}$

so $\alpha = \frac{ln(ratio_1)}{ln(2)}$

If exact integral is not known:

$I(h) - I = M\ h^{\alpha}$
$I(h/2) - I = M\ (h/2)^{\alpha}$
$I(h/4) - I = M\ (h/4)^{\alpha}$

and

$I(h) - I(h/2) = M[h^\alpha - (h/2)^\alpha]$
$I(h/2) - I(h/4) = M[(h/2)^\alpha - (h/4)^\alpha] = M\frac{1}{2^\alpha}[h^\alpha - (h/2)^\alpha]$

dividing gives:

$ratio_2 \equiv \frac{I(h)-I(h/2)}{I(h/2)-I(h/4)} = 2^\alpha$

so $\alpha = \frac{ln(ratio_2)}{ln(2)}$

Gauss Quadrature

Derivation of Gauss-3 point quadrature formula.

$\int_a^b f(x)\,dx \approx \sum_{i=1}^N [w_1 h f(x_{i-1} + r_1 h) + w_2 h f(x_{i-1} + r_2 h) + w_3 h f(x_{i-1} + r_3 h)]$

To make the algebra simpler, let's take $a = -1, b = 1, N = 1, h = 2$:

$\int_{-1}^1 f(x)\ dx \approx [2w_1 f(-1 + 2r_1) + 2w_2 f(-1 + 2r_2) + 2w_3 f(-1 + 2r_3)]$

By symmetry, we can assume:

$-1 + 2r_1 = -s, \quad 2w_1 = A$
$-1 + 2r_2 = 0, \quad 2w_2 = B$
$-1 + 2r_3 = s, \quad 2w_3 = A$

that is,

$\int_{-1}^1 f(x)\ dx \approx [Af(-s) + Bf(0) + Af(s)]$

and we try to make this formula exact for $1, x, x^2, x^3, x^4, x^5$:

$$\begin{array}{ccccccc}
2 & = & \int_{-1}^1 1dx & = & A + B + A & = & 2A + B \\
0 & = & \int_{-1}^1 xdx & = & A(-s) + B(0) + A(s) & = & 0 \\
\frac{2}{3} & = & \int_{-1}^1 x^2dx & = & A(-s)^2 + B(0)^2 + A(s)^2 & = & 2As^2 \\
0 & = & \int_{-1}^1 x^3dx & = & A(-s)^3 + B(0)^3 + A(s)^3 & = & 0 \\
\frac{2}{5} & = & \int_{-1}^1 x^4dx & = & A(-s)^4 + B(0)^4 + A(s)^4 & = & 2As^4 \\
0 & = & \int_{-1}^1 x^5dx & = & A(-s)^5 + B(0)^5 + A(s)^5 & = & 0
\end{array}$$

The solution is $s = \sqrt{0.6}, A = \frac{5}{9}, B = \frac{8}{9}$, which gives:

$r_1 = \frac{1-s}{2} = \frac{1-\sqrt{0.6}}{2} = 0.112701665$
$r_2 = \frac{1}{2} = 0.5$
$r_3 = \frac{1+s}{2} = \frac{1+\sqrt{0.6}}{2} = 0.887298335$
$w_1 = \frac{1}{2}A = \frac{5}{18}$
$w_2 = \frac{1}{2}B = \frac{8}{18}$
$w_3 = \frac{1}{2}A = \frac{5}{18}$

MATLAB function implementing Gauss 3-point formula

```
function int = gauss3(f,a,b,n)
h = (b-a)/n;
w1 = 5./18.;
w2 = 8./18.;
w3 = 5./18.;
r1 = (1-sqrt(0.6))/2.0;
r2 = 0.5;
r3 = (1+sqrt(0.6))/2.0;
int = 0.0;
for i=1:n
   xim1 = a + (i-1)*h;
   int = int + w1*h*f(xim1+r1*h) ...
             + w2*h*f(xim1+r2*h) ...
             + w3*h*f(xim1+r3*h);
end
```