Math 5330 Exam I

Name _____

1. a. If

$$A = \begin{bmatrix} 0 & 3 & 1 \\ -4 & 2 & 1 \\ 8 & 2 & 3 \end{bmatrix}$$

find a permutation matrix P, a lower triangular matrix L, and an upper triangular matrix U such that $A = PLU$.

b. What is the main use for an LU decomposition of a large matrix?

2. A MATLAB program to solve a symmetric system $Ax = b$ does most of its work in the loops:

```
for I=1:N-1
   for J=I+1:N
      for K=J:N
         A(J,K) = A(J,K) - LJI*A(I,K)
      end
   end
end
```

For large N, approximately how many multiplications are done (show work)?

3. If we use the usual finite difference approximation, the DE $u''(x) = f(x), u(0) = u(\pi) = 0$ becomes:

$U_{i+1} - 2U_i + U_{i-1} = h^2 f(x_i), \quad i = 1, ..., N-1$
$U(x_0) = U(x_N) = 0$

where $h = \pi/N, x_i = ih, U_i \approx u(x_i)$.

a. This is a linear system of $N-1$ equations for the $N-1$ unknowns $U_1, ..., U_{N-1}$. If a band solver is used to solve the system, the work is proportional to what power of N?

b. If Jacobi's iterative method is used to solve it, the iteration will take the form $U^{k+1} = BU^k + c$; what is the matrix B?

c. What are the eigenvalues of the B matrix (hint: for any $m = 1, ..., N-1$, the vector U with components $U_i = sin(mx_i)$ is an eigenvector. You will need the trig identity $sin(a+b) = sin(a)cos(b) + cos(a)sin(b)$)

d. What is the largest eigenvalue of B in absolute value? Will the Jacobi method converge?

e. Given that the error goes down each iteration by a factor approximately equal to the largest eigenvalue, estimate how many iterations of the Jacobi method are required to decrease the error by a factor of ϵ. (Hint: $cos(z) \approx 1 - z^2/2$ and $ln(1+z) \approx z$ for $z \approx 0$)

f. The total work to solve the linear system using the Jacobi iterative method is then proportional to what power of N? Which is faster for this tridiagonal system–a band solver or the Jacobi iterative method?

4. a. Find a QR decomposition of

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 12 \\ 0 & -5 \end{bmatrix}$$

b. Use this QR decomposition to find $min\|Ax - b\|_2$, where $b = (1, 2, -1)$.

c. What is the main use for a QR decomposition of a large matrix?

5. Prove that if $AA^T z = b$, and $x = A^T z$, then x minimizes $\|x\|_2$ over all solutions of $Ax = b$.

6. For what nonzero value of α is $I - \alpha\ ww^T$ orthogonal, for a vector $w \neq 0$?