MATH 5343 Cal
Final Name Key

1. Under what conditions is the following method (to approximate $u_{tt} = u_{xx}$) stable, where $U_i^k \equiv U(x_i, t_k)$ (justify answer)? (Hint: $e^{Imdx} - 2 + e^{-Imdx} = -4sin^2(m * dx/2)$)

$$\frac{U_i^{k+1} - 2U_i^k + U_i^{k-1}}{dt^2} = \frac{U_{i+1}^{k-1} - 2U_i^{k-1} + U_{i-1}^{k-1}}{dx^2}$$

always unstable

4

$$a(t_{k+1}) - 2a(t_k) + a(t_{k-1}) = \frac{dt^2}{dx^2} a(t_{k-1})\left(-4\sin^2\frac{mdx}{2}\right)$$

$$\lambda^2 - 2\lambda + (1+r) = 0 \qquad r = 4\frac{dt^2}{dx^2}\sin^2\left(\frac{mdx}{2}\right)$$

$$\lambda = 1 \pm \sqrt{r}\, i \qquad |\lambda|^2 = 1 + r > 1$$

2. a. When a Galerkin finite element method is used to approximate the solution of $u'' + f(x) = 0, u(0) = u(1) = 0$, and the approximate solution is expanded as a linear combination of piecewise linear "chapeau" trial functions $\phi_i(x) : U(x) = \sum_{i=1}^{N-1} a_i\phi_i(x)$ there results a linear system $A\mathbf{a} = \mathbf{b}$ for the unknowns $\mathbf{a} = (a_1, ..., a_{N-1})$. Give formulas (involving integrals, and only first derivatives) for the elements A_{ki}, b_k of the matrix and right hand side vector.

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$$A_{ki} = \int_0^1 \phi_k' \phi_i' \, dx$$

$$b_k = \int_0^1 f \phi_k \, dx$$

b. If the integrals in A_{ki} are calculated exactly, and a mid-point rule is used to approximate the integral in b_k, taking into account that $a_i = U(x_i)$, the resulting linear equations simplify (assuming uniformly spaced x_i) to:

4

$$\frac{-U(x_i+h) + 2U(x_i) - U(x_i-h)}{h} = \frac{h}{2}[f(x_i + h/2) + f(x_i - h/2)]$$

$$T = \frac{u(x+h) - 2u(x) + u(x-h)}{h^2} + \frac{1}{2}f\left(x + \frac{h}{2}\right) + \frac{1}{2}f\left(x - \frac{h}{2}\right)$$

$$= u'' + \frac{h^2}{12}u^{iv} + \ldots \; f + \frac{h^2}{8}f'' + \ldots = \frac{h^2}{12}u^{iv} - \frac{h^2}{8}u^{iv} \ldots$$

$$= -\frac{1}{24}h^2 u^{iv} + \ldots$$

Find the truncation error for this "finite difference" scheme.

$$T = -\frac{1}{24} h^2 u^{IV} \ldots$$

c. Explain why the collocation method cannot be used here in place of the Galerkin method.

1

$\varphi_i''(z_k) = 0$ all k

3. The usual centered, second order, finite difference approximations are used to solve $u_{xx} + u_{yy} + u_{zz} = f(x, y, z)$, with $u = 0$ on the boundary of the square $0 < x < 1, 0 < y < 1, 0 < z < 1$. If M gridlines are used in each direction, ie, $x_i = i/M, i = 0, ..., M$ and similarly for y_j, z_k, a linear system results with $(M-1)^3$ unknowns and $(M-1)^3$ equations (one equation centered on each unknown). If the unknowns are numbered by planes, that is, all unknowns on the plane z_1 are numbered first, then those on plane z_2, etc, then:

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a. Approximately how many multiplications are done if this linear system is solved by a Gauss elimination routine, which does not take any advantage of zero elements?

$$\frac{1}{3} N^3 = \frac{1}{3}(M-1)^9 \doteq \frac{1}{3} M^9$$

b. Approximately what is the half-bandwidth of the linear system? (Half-bandwidth = max $|i - j|$ such that $A_{ij} \neq 0$.)

$$L = (M-1)^2 \doteq M^2$$

c. Approximately how many multiplications are done if it is solved by a band solver, with partial pivoting?

$$2NL^2 = 2M^3 (M^2)^2 = 2M^7$$

d. Approximately how many if a band solver is used with no pivoting?

$$NL^2 \doteq M^7$$

e. Approximately how many "divisions" are done if the SOR iterative method is used with optimal ω, assuming the number of iterations is then about cM.

$$N \cdot cM = M^3 cM = cM^4$$

4. Consider the eigenvalue problem $u_{xx} + u_{yy} = \lambda u$ with $u = 0$ on the boundary of a 2D region R. An approximate eigenfunction of the form $U(x,y) = \sum_{i=1}^{N} a_i\phi_i(x,y)$ is sought, where the ϕ_i are linearly independent and each is 0 on the boundary.

a. If a Galerkin finite element method is used, a generalized eigenvalue problem $A\mathbf{a} = \lambda B\mathbf{a}$ results, where $\mathbf{a} = (a_1, ..., a_N)$. Give formulas (involving integrals, and only first derivatives) for the elements A_{ki}, B_{ki} of matrices A and B.

3

$$A_{ki} = \iint_R \nabla^2\phi_i\,\phi_k = \iint_R -\nabla\phi_i\cdot\nabla\phi_k$$

$$B_{ki} = \iint_R \phi_i\,\phi_k$$

b. Show that the matrix B in problem 4a is positive definite. (Recall that one definition of positive definite is that B is symmetric and $z^T Bz > 0$ for any nonzero vector z.) Thus, B is nonsingular.

(B is clearly symmetric)

3

$$z\cdot Bz = \sum_{k=1}^{N} z_k (Bz)_k = \sum_{k=1}^{N} z_k\left(\sum_{i=1}^{N} B_{ki} z_i\right)$$

$$= \sum_{k=1}^{N} z_k \sum_{i=1}^{N}\left(\iint_R \phi_i\phi_k\right) z_i = \iint_R \left(\sum z_k\phi_k\right)\left(\sum z_i\phi_i\right) \geq 0$$

$= 0$ only if $\sum z_i \phi_i(x) \equiv 0 \Rightarrow z_i$ all 0.

c. Write the inverse power method iteration for finding the smallest eigenvalue of $Az = \lambda Bz$ in a form where no inverses are calculated.

3

$$A z_{n+1} = B z_n \quad \leftarrow \quad z_{n+1} = (B^{-1}A)^{-1} z_n$$

d. If a collocation finite element method is used, with collocation points (x_k, y_k), a generalized eigenvalue problem $Aa = \lambda Ba$ again results; this time what are the elements of A_{ki}, B_{ki} of A and B? Is B now positive definite?

3

$$A_{ki} = \nabla^2\phi_i(x_k, y_k)$$

$$B_{ki} = \phi_i(x_k, y_k)$$

B is not even symmetric

MATH 5343 (b)
Final

Name Key

1. Under what conditions is the following method (to approximate $u_{tt} = u_{xx}$) stable, where $U_i^k \equiv U(x_i, t_k)$ (justify answer)? (Hint: $e^{Imdx} - 2 + e^{-Imdx} = -4sin^2(m * dx/2)$)

$$\frac{U_i^{k+1} - 2U_i^k + U_i^{k-1}}{dt^2} = \frac{U_{i+1}^{k+1} - 2U_i^{k+1} + U_{i-1}^{k+1}}{dx^2}$$

$r^2 = 4 \frac{dt^2}{dx^2} \sin^2\left(\frac{m dx}{2}\right)$

3

$(1 + r^2)\lambda^2 - 2\lambda + 1 = 0$

$\lambda = \frac{2 \pm \sqrt{-4r^2}}{2(1+r^2)} = \frac{1 \pm r i}{1+r^2} \qquad |\lambda|^2 = \frac{1+r^2}{(1+r^2)^2} = \frac{1}{1+r^2} < 1$

unconditionally stable

2. a. If a Galerkin finite element method is used to approximate the solution of $u^{iv} = f(x), u(0) = u(1) = u'(1) = 0, u'(0) = A$, and the approximate solution is expanded as a linear combination of trial functions $\phi_i(x) : U(x) = \Omega(x) + \sum_{i=1}^N a_i \phi_i(x)$ where $\Omega(x)$ satisfies all four boundary conditions, and each of the ϕ_k satisfies $\phi_k(0) = \phi_k'(0) = \phi_k(1) = \phi_k'(1) = 0$, there results a linear system $A\mathbf{a} = \mathbf{b}$ for the unknowns $\mathbf{a} = (a_1, ..., a_N)$. Give formulas for the elements A_{ki}, b_k of the matrix and right hand side vector.

2

$\int_0^1 u^{iv} \phi_k \, dx = \int_0^1 f \phi_k \, dx$

$[u''' \phi_k - u'' \phi_k']_0^1 + \int_0^1 u'' \phi_k'' \, dx = \int_0^1 f \phi_k \, dx$

$\sum_{i=1}^N a_i \left[\underbrace{\int_0^1 \phi_i'' \phi_k'' dx}_{A_{ki}} \right] = \underbrace{\int_0^1 f \phi_k dx - \int_0^1 \Omega'' \phi_k''}_{b_k}$

b. Same question, but now the collocation method is used.

$\Omega^{iv}(x_k) + \sum_{i=1}^N a_i \phi_i^{iv}(x_k) = f(x_k)$

2

$\sum_{i=1}^N a_i \underbrace{\phi_i^{iv}(x_k)}_{A_{ki}} = \underbrace{f(x_k) - \Omega^{iv}(x_k)}_{b_k}$

1

c. If the Galerkin method is used, how many continuous derivatives must the $\phi_i(x)$ have? If collocation is used? Can cubic Hermite basis functions be used for Galerkin? Collocation?

Galerkin $C^1(0,1)$ $\to$ yes
Collocation $C^3(0,1)$ $\to$ no

3. Suppose the ODE $u' = f(t, u)$ is approximated by:

$$11U_{n+1} - 18U_n + 9U_{n-1} - 2U_{n-2} = 6h\ f(t_{n+1}, U_{n+1})$$

2

a. Is the method stable? (Hint: $\lambda = 1$ is always a root of the characteristic polynomial, for consistent methods.)

$11\lambda^3 - 18\lambda^2 + 9\lambda - 2 = 0$ $\qquad \lambda = 1, \frac{7 \pm \sqrt{39}\,i}{22}$ $\quad |\lambda|^2 = 0.1818$

$(\lambda - 1)(11\lambda^2 - 7\lambda + 2) = 0$

yes

b. Calculate the truncation error (Hint: don't forget to normalize).

$T = \frac{1}{4} h^3 u'''$

2

2

c. Suppose this method is used to solve a linear ODE system $Au' = Bu + f(t)$, where A and B are matrices of constants (and A is nonsingular) and u is the vector of unknowns. Then every time step, a linear system must be solved which has what matrix? Suppose A and B are N by N band matrices, with half-bandwidth $L = \sqrt{N}$. Assuming N is large, the work will be what order $O(N^?)$ to solve this linear system the first time step? On every time step after the first? (Assume you take advantage of any special structure of the matrices.)

$(11A - 6hB)$

first iteration $O(NL^2) = O(N^2)$

second $O(NL) = O(N^{1.5})$

4. Write out the nonlinear overrelaxation formulas to solve the usual centered finite difference approximation to $U_{xx} + U_{yy} = U^5 + e^{x+y}$.

$$u_{ij}^{n+1} = u_{ij} - \omega \frac{\left(\frac{u_{i+1j} - 2u_{ij} + u_{i-1j}}{dx^2}\right) + \left(\frac{u_{ij+1} - 2u_{ij} + u_{i,j-1}}{dy^2}\right) - u_{ij}^5 - e^{x_i + y_j}}{\left(\frac{-2}{dx^2} - \frac{2}{dy^2} - 5u_{ij}^4\right)}$$

2

5. We want to find an eigenvalue of $\nabla^2 U = \lambda U$, with $U = 0$ on the boundary. If a finite element method is used, and U is expanded as a linear combination of basis function $\phi_k(x, y, z)$, each of which satisfies the boundary condition, a generalized eigenvalue problem $A\mathbf{a} = \lambda B\mathbf{a}$ will result.

a. What are the elements A_{ki}, B_{ki} of the A and B matrices if the Galerkin method is used? Are A and/or B symmetric?

2

$A_{ki} = \iiint -\nabla\phi_k \cdot \nabla\phi_i$ yes symmetric

$B_{ki} = \iiint \phi_k \phi_i$ yes "

b. Same questions, if the collocation method is used?

1

$A_{ki} = \nabla^2\phi_i(x_k, y_k, z_k)$ not symmetric

$B_{ki} = \phi_i(x_k, y_k, z_k)$ not "

c. Write out, in an efficient form, the shifted inverse power iteration for finding the eigenvalue of $A\mathbf{a} = \lambda B\mathbf{a}$ closest to p, and tell how this eigenvalue λ can be found from this iteration.

2

$(A - pB)\, a_{n+1} = B a_n$

$\frac{a_{n+1}}{a_n} \approx \mu \Rightarrow \lambda = \frac{1}{\mu} + p$

MATH 5343 (c)
Final

Name Key

1. We want to solve the PDE $-\nabla^2 U + U = f(x,y)$, with $\frac{\partial U}{\partial n} = -U + g(x,y)$ on the boundary. If a Galerkin finite element method is used, and U is expanded as a linear combination of basis functions $U(x,y) = \sum_{i=1}^{N} a_i \phi_i(x,y)$, a linear system of the form $A\mathbf{a} = \mathbf{b}$ will result. What are the elements A_{ki}, b_k of matrix A and vector $\mathbf{b}$. How many continuous derivatives must $\phi_i(x,y)$ have? Is A symmetric? Is A positive definite?

$$\iint -\nabla^2 U \phi_k + U\phi_k = \iint f \phi_k \qquad \phi_i \in C^0, \ A \text{ symm \& pos def.}$$

$$\int_{\partial R} -\phi_k \frac{\partial U}{\partial n} + \iint_R \sum a_i (\nabla\phi_i \cdot \nabla\phi_k + \phi_i \phi_k) = \iint f \phi_k$$

$$A_{ki} = \iint_R \nabla\phi_i \cdot \nabla\phi_k + \phi_i\phi_k + \int_{\partial R} \phi_i \phi_k \qquad b_k = \iint_R f\phi_k + \int_{\partial R} g\phi_k$$

2. We want to solve the PDE $-\nabla^2 U + U = f(x,y)$, with $U = 0$ on the boundary. If a collocation finite element method is used, and U is expanded as a linear combination of basis functions $U(x,y) = \sum_{i=1}^{N} a_i\phi_i(x,y)$, where $\phi_i(x,y) = 0$ on the boundary, a linear system of the form $A\mathbf{a} = \mathbf{b}$ will result. What are the elements A_{ki}, b_k of matrix A and vector $\mathbf{b}$. How many continuous derivative must $\phi_i(x,y)$ have? Is A symmetric?

$$A_{ki} = -\nabla^2\phi_i(x_k, y_k) + \phi_i(x_k, y_k) \qquad b_k = f(x_k, y_k)$$

$\phi_i \in C^1$ A not symmetric

3. Give two important advantages of the finite element method over finite difference methods.

a) handles non-rectangular regions better

b) handles non-uniform grids better

4. Write out, in an form where no inverses appear, the shifted inverse power iteration for finding the eigenvalue of $A\mathbf{a} = \lambda B\mathbf{a}$ closest to p, and tell how this eigenvalue λ can be found from two consecutive iterates, z_n and z_{n+1}, after convergence.

$$z_{n+1} = (B^{-1}A - pI)^{-1} z_n \qquad \lambda = p + \frac{z_n \cdot z_n}{z_n \cdot z_{n+1}}$$

$$(A - pB) z_{n+1} = B z_n$$

5. Is the following approximation consistent with $u' = f(t,u)$, and is it stable? What is the order $O(h^\alpha)$ of the error at a fixed t?
$\frac{U(t_{k+1}) - U(t_{k-2})}{3h} = \frac{1}{2} f(t_k, U(t_k)) + \frac{1}{2} f(t_{k-1}, U(t_{k-1}))$.

$T = \frac{h^3}{4} u''' + \cdots$ so consistent $\qquad \lambda^3 - 1 = 0 \qquad \lambda = 1, -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$

$|\lambda| = 1$ so stable

$\text{Error}(t) = O(h^2)$

6. a. Under what conditions is the following method (to approximate $u_{tt} = u_{xx}$) stable, where $U_i^k \equiv U(x_i, t_k)$ (justify answer)? (Hint: $e^{Imdx} - 2 + e^{-Imdx} = -4sin^2(m * dx/2)$)

$$\frac{U_i^{k+1} - 2U_i^k + U_i^{k-1}}{dt^2} = \frac{1}{3} \frac{U_{i+1}^{k-1} - 2U_i^{k-1} + U_{i-1}^{k-1}}{dx^2} + \frac{1}{3} \frac{U_{i+1}^k - 2U_i^k + U_{i-1}^k}{dx^2} + \frac{1}{3} \frac{U_{i+1}^{k+1} - 2U_i^{k+1} + U_{i-1}^{k+1}}{dx^2}$$

$$a_{k+1} - 2a_k + a_{k-1} = \frac{dt^2}{dx^2} \frac{1}{3} \left(-4 \sin^2 \frac{mdx}{2}\right)(a_{k-1} + a_k + a_{k+1})$$

$$\equiv -r(a_{k-1} + a_k + a_{k+1})$$

$$(1+r)\lambda^2 + (-2+r)\lambda + (1+r) = 0$$

$$\lambda = \frac{2-r \pm \sqrt{3r(r+4)}\, i}{2(1+r)} \qquad |\lambda|^2 = \left(\frac{2-r}{2(1+r)}\right)^2 + \frac{3r(r+4)}{(2(1+r))^2} = 1$$

always stable

b. What is the order $O(dt^n) + O(dx^m)$ of the truncation error? You can just guess!

$O(dt^2) + O(dx^2)$