1. Under what conditions is the following method (to approximate  $u_{tt} = u_{xx}$ ) stable, where  $U_i^k \equiv U(x_i, t_k)$  (justify answer)? (Hint:  $e^{Imdx} - 2 + e^{-Imdx} = 0$  $-4sin^{2}(m*dx/2))$ 

$$\frac{U_i^{k+1} - 2U_i^k + U_i^{k-1}}{dt^2} = \frac{U_{i+1}^{k-1} - 2U_i^{k-1} + U_{i-1}^{k-1}}{dx^2}$$
 always untable

r= 4 of sin 2 max

$$a(t_{n+1}) - 2a(t_n) + a(t_{n+1}) = \frac{dt^2}{dx^2} a(t_{n+1})[-4\sin^2 \frac{mdx}{2}]$$
  
 $\lambda^2 - 2\lambda + (1+r) = 0$   $r = 4 \frac{dx}{dx}$   
 $\lambda = 1 \pm 5ri |\lambda|^2 = 1 + r > 1$ 

a. When a Galerkin finite element method is used to approximate the solution of u'' + f(x) = 0, u(0) = u(1) = 0, and the approximate solution is expanded as a linear combination of piecewise linear "chapeau" trial functions  $\phi_i(x):U(x)=\sum_{i=1}^{N-1}a_i\phi_i(x)$  there results a linear system  $A\mathbf{a} = \mathbf{b}$  for the unknowns  $\mathbf{a} = (a_1, ..., a_{N-1})$ . Give formulas (involving integrals, and only first derivatives) for the elements  $A_{ki}$ ,  $b_k$  of the matrix and right hand side vector.

$$A_{ki} = S_0' f_k' f_i' dx$$

$$b_k = S_0' f f_k dx$$

b. If the integrals in  $A_{ki}$  are calculated exactly, and a mid-point rule is used to approximate the integral in  $b_k$ , taking into account that  $a_i = U(x_i)$ , the resulting linear equations simplify (assuming uniformly spaced  $x_i$ ) to:

$$\frac{-U(x_i+h)+2U(x_i)-U(x_i-h)}{h} = \frac{h}{2}[f(x_i+h/2)+f(x_i-h/2)]$$

$$T = \frac{u(x+h)-2u(x)+u(x-h)}{h^2} + \frac{1}{2}f(x+\frac{1}{2}) + \frac{1}{2}f(x-\frac{1}{2})$$

$$= u'' + \frac{h^2}{12}u'' + \dots + \frac{h^2}{8}f'' + \dots = \frac{h^2}{12}u'' - \frac{h^2}{8}u''$$

$$= (-\frac{1}{24}h^2u'' + \dots + \frac{1}{8}f'' + \dots + \frac{1}{8}f''$$

Find the truncation error for this "finite difference" scheme.

c. Explain why the collocation method cannot be used here in place of the Galerkin method.

4."(z4) =0 all K

- 3. The usual centered, second order, finite difference approximations are used to solve  $u_{xx} + u_{yy} + u_{zz} = f(x,y,z)$ , with u=0 on the boundary of the square 0 < x < 1, 0 < y < 1, 0 < z < 1. If M gridlines are used in each direction, ie,  $x_i = i/M, i = 0, ..., M$  and similarly for  $y_j, z_k$ , a linear system results with  $(M-1)^3$  unknowns and  $(M-1)^3$  equations (one equation centered on each unknown). If the unknowns are numbered by planes, that is, all unknowns on the plane  $z_1$  are numbered first, then those on plane  $z_2$ , etc, then:
  - a. Approximately how many multiplications are done if this linear system is solved by a Gauss elimination routine, which does not take any advantage of zero elements?  $\frac{1}{3} N^3 = \frac{1}{3} (M-1)^9 \stackrel{?}{=} \frac{1}{3} M^9$
  - b. Approximately what is the half-bandwidth of the linear system? (Half-bandwidth = max |i-j| such that  $A_{ij} \neq 0$ .)

L=(M-1)2 7 M2

c. Approximately how many multiplications are done if it is solved by a band solver, with partial pivoting?

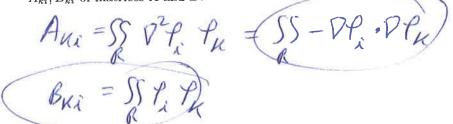
 $2NL^2 = 2M^3(M^2)^2 = 2M^7$ 

d. Approximately how many if a band solver is used with no pivoting?  $\mathcal{N}^{2}$ 

e. Approximately how many "divisions" are done if the SOR iterative method is used with optimal  $\omega$ , assuming the number of iterations is then about cM.

N.c.M = M3cM = CM4

- 4. Consider the eigenvalue problem  $u_{xx} + u_{yy} = \lambda u$  with u = 0 on the boundary of a 2D region R. An approximate eigenfunction of the form  $U(x,y) = \sum_{i=1}^{N} a_i \phi_i(x,y)$  is sought, where the  $\phi_i$  are linearly independent and each is 0 on the boundary.
  - a. If a Galerkin finite element method is used, a generalized eigenvalue problem  $A\mathbf{a} = \lambda B\mathbf{a}$  results, where  $\mathbf{a} = (a_1, ..., a_N)$ . Give formulas (involving integrals, and only first derivatives) for the elements  $A_{ki}$ ,  $B_{ki}$  of matrices A and B.



Show that the matrix B in problem 4a is positive definite. (Recall that one definition of positive definite is that B is symmetric and  $z^TBz > 0$  for any nonzero vector z.) Thus, B is nonsingular. b. Show that the matrix B in problem 4a is positive definite. (Recall

$$z^TBz > 0$$
 for any nonzero vector  $z$ .) Thus,  $B$  is nonsingular.  
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 $z^TBz > 0$  for any nonzero vector  $z$ .) Thus,  $B$  is nonsingular.  
 $z^TBz > 0$  for any nonzero vector  $z$ .) Thus,  $z^TBz > 0$  for  $z^TBz > 0$  fo

eigenvalue of  $Az = \lambda Bz$  in a form where no inverses are calculated.

is not even

d. If a collocation finite element method is used, with collocation points  $(x_k, y_k)$ , a generalized eigenvalue problem  $Aa = \lambda Ba$  again results; this time what are the elements of  $A_{ki}$ ,  $B_{ki}$  of A and B? Is B now positive definite?

$$A_{ui} = \mathcal{P}^2 f_i(x_u, y_k)$$

$$B_{ui} = f_i^3(x_u, y_k)$$

MATH 5343 (6)

1. Under what conditions is the following method (to approximate  $u_{tt} = u_{xx}$ ) stable, where  $U_i^k \equiv U(x_i, t_k)$  (justify answer)? (Hint:  $e^{Imdx} - 2 + e^{-Imdx} =$  $-4sin^{2}(m*dx/2)$ 

 $\frac{U_i^{k+1} - 2U_i^k + U_i^{k-1}}{dt^2} = \frac{U_{i+1}^{k+1} - 2U_i^{k+1} + U_{i-1}^{k+1}}{dt^2}$ 

r= 4 dt sx 2 (mdx)

 $(/+ r^2) \lambda^2 - 2\lambda + 1 = 0$ 

 $\lambda = \frac{2 \pm \sqrt{-4r^2}}{2(1+r^2)} = \frac{1 \pm ri}{1 + r^2} |\lambda|^2 = \frac{1}{(1+r^2)^2} = \frac{1}{(1+r^2)^2} = \frac{1}{(1+r^2)^2}$ 

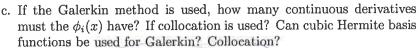
unconditionally stable

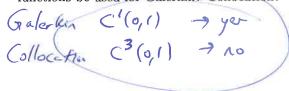
a. If a Galerkin finite element method is used to approximate the solution of  $u^{iv} = f(x), u(0) = u(1) = u'(1) = 0, u'(0) = A$ , and the approximate solution is expanded as a linear combination of trial functions  $\phi_i(x): U(x) = \Omega(x) + \sum_{i=1}^N a_i \phi_i(x)$  where  $\Omega(x)$  satisfies all four boundary conditions, and each of the  $\phi_k$  satisfies  $\phi_k(0) = \phi'_k(0) = \phi_k(1) = \phi'_k(1) = 0$ , there results a linear system  $A\mathbf{a} = \mathbf{b}$  for the unknowns  $\mathbf{a} = (a_1, ..., a_N)$ . Give formulas for the elements  $A_{ki}$ ,  $b_k$  of the matrix and right hand side vector.

Surpalx = Stplx [u""Px - u"P"] + 5' u"P" 2x = 5 f Px dx Ea: Stradx - S'S"Put Bu

b. Same question, but now the collocation method is used.

 $\int_{-\infty}^{\infty} (x_k) + \sum_{k=0}^{\infty} q_k f_k'(x_k) = f(x_k)$ Z a PIV(X) = f(x) - six





3. Suppose the ODE u' = f(t, u) is approximated by:

$$11U_{n+1} - 18U_n + 9U_{n-1} - 2U_{n-2} = 6h \ f(t_{n+1}, U_{n+1})$$

a. Is the method stable? (Hint:  $\lambda=1$  is always a root of the characteristic polynomial, for consistent methods.)

$$(1/\lambda^{3}-18\lambda^{2}+9\lambda-2=0)$$
  $\lambda=(1,7\pm \sqrt{39}x)(\lambda)^{2}=0.1818$   
 $(\lambda-1)(1/\lambda^{2}-7\lambda+2)=0$  yer

b. Calculate the truncation error (Hint: don't forget to normalize).

c. Suppose this method is used to solve a linear ODE system Au' = Bu + f(t), where A and B are matrices of constants (and A is non-singular) and u is the vector of unknowns. Then every time step, a linear system must be solved which has what matrix? Suppose A and B are N by N band matrices, with half-bandwidth  $L = \sqrt{N}$ . Assuming N is large, the work will be what order  $O(N^2)$  to solve this linear system the first time step? On every time step after the first? (Assume you take advantage of any special structure of the matrices.)

(11A-GhB)

First iteration 
$$O(NL^2) = O(N^2)$$

Second  $O(NL) = O(N^{65})$ 

2

2

4. Write out the nonlinear overrelaxation formulas to solve the usual centered finite difference approximation to  $U_{xx} + U_{yy} = U^5 + e^{x+y}$ .

 $u_{ij} - \omega = \frac{(u_{xij} - 2u_{xj} + u_{xij})}{(u_{xj} - 2u_{xj} + u_{xij})} - u_{xi}^{5} - e^{x_{i} + y_{i}}$   $\left(-\frac{2}{2x_{i}} - \frac{2}{2x_{i}} - Su_{ij}^{5}\right)$ 

- 5. We want to find an eigenvalue of  $\nabla^2 U = \lambda U$ , with U = 0 on the boundary. If a finite element method is used, and U is expanded as a linear combination of basis function  $\phi_k(x, y, z)$ , each of which satisfies the boundary condition, a generalized eigenvalue problem  $A\mathbf{a} = \lambda B\mathbf{a}$  will result.
  - a. What are the elements  $A_{ki}, B_{ki}$  of the A and B matrices if the Galerkin method is used? Are A and/or B symmetric?

Ahi = SSS - DP PP yer symmetric Bui = SSS Puti yer

b. Same questions, if the collocation method is used?

$$A_{ni} = \nabla^2 f(x_n, y_n, z_n) \quad \text{not symmetric}$$

$$B_{Ki} = f(x_n, y_n, z_n) \quad \text{not} \quad \text{not}$$

c. Write out, in an efficient form, the shifted inverse power iteration for finding the eigenvalue of  $Aa = \lambda Ba$  closest to p, and tell how this eigenvalue  $\lambda$  can be found from this iteration.

(A-pB) an = Ban a<sub>AHI</sub>  $\approx \mu$   $\Rightarrow$ 

MATH 5343 ( Final	(c)	Name	Key	 
			0	

1. We want to solve the PDE  $-\nabla^2 U + U = f(x,y)$ , with  $\frac{\partial U}{\partial n} = -U + g(x,y)$  on the boundary. If a Galerkin finite element method is used, and U is expanded as a linear combination of basis functions  $U(x,y) = \sum_{i=1}^{N} a_i \phi_i(x,y)$ , a linear system of the form Aa = b will result. What are the elements  $A_{ki}, b_k$  of matrix A and vector b. How many continuous derivatives must  $\phi_i(x,y)$  have? Is A symmetric? Is A positive definite?

$$SS - D^2u f_u + Uf_u = SSf f_u \qquad f_i \in C^{\circ}, A symb for dof.$$

$$S - f_u \stackrel{2u}{\Rightarrow} n + SS \stackrel{2u}{\Rightarrow} (Df_i \cdot Df_u + f_i f_u) = SSf f_u$$

$$Au = SSDf_i \cdot Df_u + f_i f_u + Sf f_u \qquad f_u = SSf f_u + Sg f_u$$

2. We want to solve the PDE  $-\nabla^2 U + U = f(x,y)$ , with U = 0 on the boundary. If a collocation finite element method is used, and U is expanded as a linear combination of basis functions  $U(x,y) = \sum_{i=1}^{N} a_i \phi_i(x,y)$ , where  $\phi_i(x,y) = 0$  on the boundary, a linear system of the form Aa = b will result. What are the elements  $A_{ki}$ ,  $b_k$  of matrix A and vector b. How many continuous derivative must  $\phi_i(x,y)$  have? Is A symmetric?

$$A_{ii} = -v^2 f_i(x_{ii}y_{ii}) + f_i(x_{ii}y_{ii}) \quad b_{ii} = f(x_{ii}y_{ii})$$

$$f_i \in C' \quad A_{ii} = f(x_{ii}y_{ii}) \quad b_{ii} = f(x_{ii}y_{ii})$$

3. Give two important advantages of the finite element method over finite difference methods.

a) handle non-unistam gride better

4. Write out, in an form where no inverses appear, the shifted inverse power iteration for finding the eigenvalue of  $A\mathbf{a} = \lambda B\mathbf{a}$  closest to p, and tell how this eigenvalue  $\lambda$  can be found from two consecutive iterates,  $z_n$  and  $z_{n+1}$ ,

$$\xi_{ne_1} = (B^{\dagger}A - pI)^{\dagger} \xi_n$$

$$(A - pB) \xi_{ne_1} = B \xi_n$$

5. Is the following approximation consistent with u' = f(t, u), and is it stable? What is the order  $O(h^{\alpha})$  of the error at a fixed t?  $\frac{U(t_{k+1})-U(t_{k-2})}{\frac{2}{h}} = \frac{1}{2}f(t_k, U(t_k)) + \frac{1}{2}f(t_{k-1}, U(t_{k-1})).$ 

$$Emor(H) = O(L^2)$$

a. Under what conditions is the following method (to approximate  $u_{tt} =$  $u_{xx}$ ) stable, where  $U_i^k \equiv U(x_i, t_k)$  (justify answer)? (Hint:  $e^{Imdx} - 2 + e^{-Imdx} = -4sin^2(m*dx/2)$ )

$$\frac{U_i^{k+1} - 2U_i^k + U_i^{k-1}}{dt^2} = \frac{1}{3} \frac{U_{i+1}^{k-1} - 2U_i^{k-1} + U_{i-1}^{k-1}}{dx^2} + \frac{1}{3} \frac{U_{i+1}^{k} - 2U_i^k + U_{i-1}^k}{dx^2} + \frac{1}{3} \frac{U_{i+1}^{k+1} - 2U_i^{k+1} + U_{i-1}^{k+1}}{dx^2}$$

$$\lambda = \frac{2-r \pm \sqrt{3r(n+r)}}{2(1+r)}$$

b. What is the order  $O(dt^n) + O(dx^m)$  of the truncation error? You can just guess!