MATH 5343 (a)
Test I

Name Key

1. True or False:

10

a. If the LU decomposition of a tridiagonal matrix is done using partial pivoting, the resulting matrices L and U are also tridiagonal. False

b. The backwards (implicit) Euler method
$\frac{U_{k+1}-U_k}{h} = f(t_{k+1}, U_{k+1})$
is much better suited for stiff problems than the forwards (explicit) Euler method. True

c. All Adams methods (for $u' = f(t,u)$) are stable. True

d. It is easier to vary the stepsize h for Runge-Kutta methods than for multistep methods. True

e. An equation of third degree ($u''' = f(t, u, u', u'')$) can be reduced to a system of three first order equations. True

f. The number of multiplications required to solve an N by N full system of linear equations using Gaussian elimination is $\frac{1}{3}N^2$. False

g. It is possible to design explicit multistep methods which work well on stiff systems. False

h. For stable methods, the error is of the same order as the truncation order. True

i. If an ODE solver produces an error of 10^{-3} at $t = 1$, when $h = 0.01$, and an error of 10^{-6} at $t = 1$, when $h = 0.001$, then the experimental global error is $O(h^2)$. False

j. The amount of work required to solve an N by N tridiagonal system using a band solver is $O(N)$. True

2. a. Is the following method stable? (Justify answer)

3-3-1-1 = 8

$$\frac{U_{k+1}+4U_k-5U_{k-1}}{6h} = \frac{2}{3}f(t_k, U_k) + \frac{1}{3}f(t_{k-1}, U_{k-1})$$

$\lambda^2 + 4\lambda - 5 = 0 \qquad \lambda = 1, -5$ unstable

b. Is the method consistent? (Justify answer)

$$T = \frac{u(t+h) + 4u(t) - 5u(t-h)}{6h} - \frac{2}{3}u'(t) - \frac{1}{3}u'(t-h) = \frac{h^3 u^{iv}}{36}$$ yes

c. Is this a backward difference method? no

d. Is this an Adams method? no

3. a. Find the optimal values for A and B in the approximation (to $u' = f(t,u)$)

$$\frac{U_{k+1}-U_k}{h} = Af(t_k, U_k) + Bf(t_{k-1}, U_{k-1})$$

$6 = 3-2-1$

$$T = \frac{u(t+h)-u(t)}{h} - Au'(t) - Bu'(t-h)$$

$$= u'(1-A-B) + hu''\left(\tfrac{1}{2}+B\right) + h^2u'''\left(\tfrac{1}{6}-\tfrac{1}{2}B\right)+\ldots$$

$$A = \tfrac{3}{2}, \quad B = -\tfrac{1}{2}$$

b. What is then the truncation error?

$$T = \left(\tfrac{1}{6} - \tfrac{1}{2}\left(-\tfrac{1}{2}\right)\right)h^2u''' + \ldots = \tfrac{5}{12}h^2u'''$$

c. Is the method implicit or explicit? explicit

4. For the problem $u' = -100u, u(0) = 1$, which has exact solution $u(t) = e^{-100t}$,

6

a. Compute the Euler method approximation to $u(1)$ using $h = 0.1$. For what values of h does Euler's method produce a good approximation?

$$U_{n+1} = U_n + h(-100u_n) = (1-100h)u_n$$

$$u_n = (1-100h)^n \qquad h = 0.1 \Rightarrow u_{10} = (-9)^{10}$$

good when $h \leq .02$

b. Compute the backward Euler method approximation to $u(1)$ using $h = 0.1$. For what values of h does backward Euler produce a good approximation?

$$U_{n+1} = U_n + h(-100u_{n+1})$$

$$(1+100h)u_{n+1} = u_n \qquad h = 0.1 \Rightarrow u_{10} = \left(\tfrac{1}{11}\right)^{10}$$

$$u_n = \left(\frac{1}{1+100h}\right)^n$$

good for all h

MATH 5343 (6) Name Key
Test I

1. True or False:

T a. Once the LU decomposition of a matrix A is known, a system $Ax = b$ can be solved in $O(N^2)$ operations.

T b. All Adams methods (for $u' = f(t, u)$) are stable.

F c. When a Runge-Kutta method of high order is used, a different method must be used to take the first few steps.

F d. It is possible to design explicit multistep methods which work well on stiff systems.

T e. If an ODE solver produces an error of 10^{-3} at $t = 1$, when $h = 0.01$, and an error of 10^{-7} at $t = 1$, when $h = 0.001$, then the experimental global error is $O(h^4)$.

T f. If a matrix is diagonal dominant and symmetric, then partial pivoting means no pivoting.

4.5

F g. If an N by N band matrix A has half-bandwidth L, where $1 \ll L \ll N$, a system $Ax = b$ can be solved in $O(NL)$ operations using a band solver.

F h. When using a multistep method of $O(h^4)$ and a Taylor series method to take the first few steps, a Taylor polynomial of degree at least 4 should be used, to preserve the order of the multistep method.

T i. If you need to solve several linear systems with the same *banded* matrix, but different right hand sides, it is more efficient to save the LU decomposition when solving the first system, and use this to solve the other systems, than to find the inverse and multiply each right hand side by the inverse matrix.

2. a. Is the following method stable? (Justify answer)

$$\frac{U_{k+1} - U_{k-2}}{3h} = \frac{1}{2} f(t_k, U_k) + \frac{1}{2} f(t_{k-1}, U_{k-1})$$

$\lambda^3 - 1 = 0 \quad \lambda = 1, -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$ so stable

2

b. Find the truncation error, and tell if the method is consistent or not.

$$T = \frac{(u + hu' + \frac{h^2}{2}u'' + \frac{h^3}{6}u''' + \ldots) - (u - 2hu' + \frac{4h^2}{2}u'' - \frac{8h^3}{6}u''' \ldots)}{3h}$$

3

$$- \frac{1}{2}u' - \frac{1}{2}(u' - hu'' + \frac{h^2}{2}u''' \ldots) = \frac{1}{4}h^2 u''' + \ldots$$

Consistent

3. a. Find the optimal values for A and B in the approximation (to $u' = f(t, u)$)

$$\frac{U_{k+1}-U_k}{h} = Af(t_{k+1}, U_{k+1}) + Bf(t_k, U_k)$$

3

$$T = \frac{\left(u + hu' + \frac{h^2}{2}u'' + \frac{h^3}{6}u''' \ldots\right) - u}{h} - A\left(u' + hu'' + \frac{h^2}{2}u''' \ldots\right) - Bu'$$

$A = \frac{1}{2}$

$B = \frac{1}{2}$

$$= u'(1-A-B) + hu''\left(\tfrac{1}{2} - A\right) + h^2u'''\left(\tfrac{1}{6} - A\tfrac{1}{2}\right) + \ldots$$

b. What is then the truncation error?

2

$$T = h^2u'''\left(\tfrac{1}{6} - \tfrac{1}{4}\right) = -\frac{h^2}{12}u'''$$

c. Is the method implicit or explicit?

1

4. The problem $u' = -au, u(0) = 1$, where $a > 0$, has exact solution $u(t) = e^{-at}$, which decreases with time.

a. For what (positive) values of h does the Euler method produce a solution which decreases (in absolute value) with time?

$$\frac{u_{n+1} - u_n}{h} = -au_n \qquad u_{n+1} = (1-ah)u_n \qquad 1 - ah > -1$$

$$h < \frac{2}{a}$$

4.5

b. Same question as (a) for backward Euler method.

$$\frac{u_{n+1} - u_n}{h} = -au_{n+1} \qquad u_{n+1} = \frac{1}{1+ah}u_n \qquad \text{all } h > 0$$

c. Same question as (a) for the method of problem 3.

$$\frac{u_{n+1} - u_n}{h} = -\frac{1}{2}au_{n+1} - \frac{1}{2}au_n$$

$$u_{n+1} = \frac{\left(1 - \frac{h}{2}a\right)}{\left(1 + \frac{ha}{2}\right)}u_n \qquad \text{all } h > 0$$

MATH 5343 (c)
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1. True or False:

T a. If Gaussian elimination is done on a large symmetric matrix and no pivoting is done, the work can be cut by approximately half by taking advantage of symmetry.

T b. Partial pivoting is equivalent to no pivoting, if the matrix is diagonal dominant and symmetric.

F c. All backward difference methods are stable.

T d. If two linear systems are solved with the same banded matrix, of size N by N and bandwidth k ($1 << k << N$), using Gaussian elimination, the first requires $O(Nk^2)$ work, but if the LU decomposition is saved, the second requires only $O(Nk)$ work.

F e. If an ODE solver produces an error of 10^{-6} at $t = 1$, when $h = 0.1$, and an error of 10^{-12} at $t = 1$, when $h = 0.001$, then the experimental global error is $O(h^2)$.

T f. Adaptive methods for ODEs decide on a stepsize by trying two different methods, or two different stepsizes, and comparing the results each step.

F g. It is possible to design explicit multistep methods which work well on stiff systems.

F h. The ODE system $y' = v, v' = -10000y$ is stiff.

F i. Forward Euler is better suited for stiff ODE problems than backward Euler.

F j. If you need to solve several systems with the same band matrix, it is more efficient to form the inverse and multiply each right hand side by the inverse, than to use the LU decomposition.

3

2. a. Is the following method stable? (Justify answer)

$$2U_{k+1} + 3U_k - 6U_{k-1} + U_{k-2} = 6hf(t_k, U_k)$$

$$2\lambda^3 + 3\lambda^2 - 6\lambda + 1 = 0$$

$$(\lambda - 1)(2\lambda^2 + 5\lambda - 1) = 0$$

$$\lambda = 1, -2.686, 0.186$$

unstable

1

b. Is the method consistent? (Justify answer, but don't necessarily have to find the truncation error.)

$$T = \frac{2u(t+h) + 3u(t) - 6u(t-h) + u(t-2h)}{6h} - u'(t)$$

3 $$= \frac{h^3 u^{iv}}{12} \quad \text{so yes} \quad \left(\text{just need to show } \tfrac{1}{h} \text{ and } 1 \text{ term } 0\right)$$

c. Is the method explicit?

1 yes

3. a. Find the optimal values for A, B, C in the approximation (to $u' = f(t,u)$)

$$\frac{U_{k+1} - U_{k-1}}{2h} = Af(t_{k+1}, U_{k+1}) + Bf(t_k, U_k) + Cf(t_{k-1}, U_{k-1})$$

$$T = \frac{u(t+h) - u(t-h)}{2h} - Au'(t+h) - Bu'(t) - Cu'(t-h)$$

3 $$= u'(1 - A - B - C) + hu''(C - A) + h^2 u'''\left(\frac{1}{6} - \frac{A}{2} - \frac{C}{2}\right) + \frac{h^3 u^{iv}}{6}(C - A) + \frac{h^4 u^{v}}{24}\left(\frac{1}{5} - A - C\right) \ldots$$

$$A + B + C = 1$$
$$-A + C = 0$$
$$A + C = \frac{1}{3}$$
$$\Rightarrow A = \frac{1}{6} = C$$
$$B = \frac{4}{6}$$

b. Is the method stable (Justify answer)?

2 $\lambda^2 - 1 = 0 \quad \lambda = 1, -1 \quad$ yes

4. Consider the general multistep method:

$$\frac{U(t_{k+1}) + \alpha_1 U(t_k) + \alpha_2 U(t_{k-1}) + \cdots + \alpha_m U(t_{k+1-m})}{h}$$

$$= \beta_0 f(t_{k+1}, U(t_{k+1})) + \cdots + \beta_m f(t_{k+1-m}, U(t_{k+1-m}))$$

a. Explain how the coefficients α_i, β_i are found for the Adams-Bashforth (explicit) methods, in general terms.

3 $\alpha_1 = -1$ $\alpha_2 .. \alpha_m = 0$, $\beta_0 = 0$, other β_i chosen to max truncation error order

b. Explain how the coefficients α_i, β_i are found for the Adams-Moulton (implicit) methods.

3 $\alpha_1 = -1$ $\alpha_2 ... \alpha_m = 0$, β_i chosen to max truncation error order

c. Explain how the coefficients α_i, β_i are found for the backward difference methods.

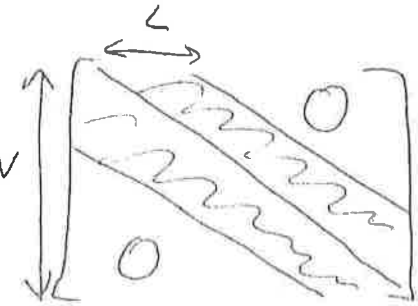
3 $\beta_1 .. \beta_m = 0$ $\beta_0, \alpha_1 .. \alpha_m$ chosen to maximize truncation error order

MATH 3543
Test I

Name: Key

1. a) If an N by N matrix A has bandwidth L

3 $\left(L \equiv \max_{A_{ij} \neq 0} |i-j| \right)$



approximately how many multiplications are required to solve $Ax=b$ using Gaussian elimination *without* pivoting? Assume $1 \ll L \ll N$.

$\boxed{NL^2}$

2 b) Approx. how many multiplications to do the back substitution?

$\boxed{NL}$

2. If $\dfrac{U_{n+1} - U_{n-1}}{2h} = f(t_n, U_n)$ is used to approximate $u' = f(t,u)$;

2 a) Is the method stable? (verify answer)

$\lambda^2 - 1 = 0 \quad \lambda = \pm 1$ yes

3 b) What is the truncation error?

$\boxed{\frac{h^2}{6} u'''(\xi_n)}$

$$T = \frac{u(t+h) - u(t-h)}{2h} - u'(t)$$

$$= \frac{\left[\cancel{u} + hu' + \cancel{\tfrac{h^2}{2}u''} + \tfrac{h^3}{6}u''' + \dots\right] - \left[\cancel{u} - hu' + \cancel{\tfrac{h^2}{2}}u'' - \tfrac{h^3}{6}u''' + \dots\right]}{2h} - u'$$

$$= \frac{2hu' + \frac{h^3}{3}u''' + \dots}{2h} - u' = \frac{h^2}{6}u''' + \dots$$

3. The exact solution to $\begin{cases} u' = au \\ u(0) = 1 \end{cases}$ is $u(t) = e^{at}$.

$$\frac{U_{n+1} - U_n}{h} = a U_n \qquad U_{n+1} = (1 + ah) U_n \qquad U_n = (1+ah)^n$$

2 a) If $a > 0$, the exact solution obviously increases with t. For what values of h does the Euler method approximation also increase with t?

$\boxed{\text{all } h > 0}$

2 b) If $a < 0$, the exact solution decreases with t. For what values of h does the Euler method approximation also decrease in abs. value with t?

$1 + ah > -1 \qquad h < \frac{-2}{a} \qquad \boxed{h < \frac{-2}{a}}$

1 c) How does your answer to (b) relate to the idea of "absolute stability"?

Euler decrease with time $\Leftrightarrow$ ah in region of abs. stabi.

5 4. If Euler's method is used to solve $\begin{cases} u' = f(t) \\ u(0) = A \end{cases}$:

$$\frac{U_{n+1} - U_n}{h} = f(t_n) \qquad U_0 = A$$

show that Euler's method converges.

Hint: find a finite difference equation satisfied by the exact solution, involving the truncation error, and one satisfied by the error $e(t_n) \equiv U_n - u(t_n)$. Then get a bound on $e(t_n)$ in terms of the truncation error, $T_n = \frac{h}{2} u''(\xi_n)$

$$\frac{u_{n+1} - u_n}{h} = f(t_n) + T_n$$

$$\frac{e_{n+1} - e_n}{h} = -T_n \qquad e_0 = 0 \qquad e_{n+1} = e_n - h T_n$$

$$|e_n| = \left| -h \left[T_1 + T_2 + \dots T_n \right] \right| \le h n T_{max}$$