MATH 5343 (a)
Test II

Name Key

1. Consider the method ($U_i^k = U(x_i, t_k)$)

$$\frac{U_i^{k+1}-U_i^k}{dt} = (1-\alpha)\frac{U_{i+1}^k-2U_i^k+U_{i-1}^k}{dx^2} + \alpha\frac{U_{i+1}^{k+1}-2U_i^{k+1}+U_{i-1}^{k+1}}{dx^2}$$

(*) (assume $\alpha > 0$) 1

a. For what value of α is the truncation error of highest order, when this is used to approximate $u_t = u_{xx}$? (no need to justify answer)

$\alpha = \frac{1}{2}$

4 b. Use the Fourier method to analyze stability, that is, plug in $U_i^k = a(t_k)e^{Imx_i}$. For what values of α is this method unconditionally stable? (Hint: $e^{Imdx} - 2 + e^{-Imdx} = -4sin^2(m*dx/2)$)

$r = \frac{4dt}{dx^2}\sin^2\left(\frac{mdx}{2}\right) \geq 0$

$$\lambda = \frac{1-(1-\alpha)r}{1+\alpha r} = 1 - \frac{r}{1+\alpha r}$$

for stability $1 - \frac{r}{1+\alpha r} > -1$

$r(1-2\alpha) < 2$

if $\alpha \geq \frac{1}{2}$ always stable

if $\alpha < \frac{1}{2}$ stable if $r < \frac{2}{1-2\alpha}$

1 c. For what values of α is the method implicit? $\alpha \neq 0$

4 2. Write a second order, centered difference approximation to the PDE $u_{xx} + u_{yy} + yu_x + 2u = sin(x)$ and calculate its truncation error.

$$\frac{U_{i+1,j} - 2U_{ij} + U_{i-1,j}}{dx^2} + \frac{U_{i,j+1} - 2U_{ij} + U_{i,j-1}}{dy^2} + y_j\frac{U_{i+1,j} - U_{i-1,j}}{2dx} + 2U_{ij} = \sin(x_i)$$

$$T = \frac{dx^2}{12}u_{xxxx} + \frac{dy^2}{12}u_{yyyy} + y\frac{dx^2}{6}u_{xxx} + \dots$$

3. Consider the boundary value problem $u^{iv}(x) = f(x)$.

3 a. Develop a second order finite difference approximation by starting from the approximation:

$$u^{iv}(x_i) \approx \frac{u''(x_{i+1})-2u''(x_i)+u''(x_{i-1})}{dx^2}$$

and replace each second derivative by a centered difference approximation.

$$U_{i+2} - 4U_{i+1} + 6U_i - 4U_{i-1} + U_{i-2} = f(x_i)(dx)^4$$

3 b. Write out the Successive Overrelaxation iteration, for solving this system of equations.

$$U_i^{K+1} = U_i^K - \frac{\omega}{6}\left[U_{i+2}^K - 4U_{i+1}^K + 6U_i^K - 4U_{i-1}^{K+1} + U_{i-2}^{K+1} - dx^4 f(x_i)\right]$$

1 c. Which method will solve this system faster, a band solver or SOR? band solver

4. Consider the damped wave equation
$u_{tt} + au_t = c^2u_{xx} + f(x,t)$
3 with initial conditions
$u(x,0) = h(x)$
$u_t(x,0) = g(x)$

An explicit, centered finite difference approximation will require starting values at the first two time levels, that is, we need to know $u(x,0)$ and $u(x,dt)$. Obviously we can take $u(x,0) = h(x)$; suggest a reasonable formula to use for $u(x,dt)$, which will preserve the second order accuracy of the approximation.

$$u(x,dt) \cong u(x,0) + u_t(x,0)dt + \frac{1}{2}u_{tt}(x,0)dt^2$$
$$= h(x) + g(x)dt + \frac{dt^2}{2}\left[-ag(x) + c^2h''(x) + f(x,0)\right]$$

10

5. True or False:

a. It is impossible to design explicit methods which are unconditionally stable and consistent with the heat equation ($u_t = Du_{xx}$). True

b. The stability criteria for the usual explicit finite difference approximation to the wave equation ($u_{tt} = c^2 u_{xx}$) is less severe than the stability criterion for the usual explicit finite difference approximation to the heat equation, hence explicit methods are more popular for approximating the wave equation than the heat equation. True

c. If the exact solution of a PDE at a point (x, t) depends on the initial conditions over a certain interval, while, as dt and $dx \to 0$, the finite difference approximations at that point all depend on the initial conditions over a proper subset of this interval, the finite difference method cannot be stable (assume it is consistent). True

d. Same as (c) but replace "proper subset" with "proper superset". False

e. For the shifted inverse power method, choosing the shift parameter closer to an eigenvalue will make the method converge more rapidly to this eigenvalue, generally. True

f. Nonlinear successive overrelaxation, with $\omega = 1$, is equivalent to Newton's method with the Jacobian matrix approximated by its diagonal part. False

g. For the transport (convection only) equation, the boundary conditions should be specified "downwind". False

h. The explicit upwind approximation to the transport equation is unconditionally stable. False

i. If the initial conditions are discontinuous, the solution to the diffusion/convection equation $u_t = Du_{xx} - vu_x$ is continuous everywhere for $t > 0$, assuming $D > 0$; but if $D = 0$ (ie, there is now no diffusion, only convection) the solution may still be discontinuous for $t > 0$. True

j. If the shifted inverse power method is used to find an eigenvalue of a matrix, with fixed shift, the LU decomposition found on the first iteration can be used on subsequent iterations to decrease the computer time for these iterations. True

MATH 5343 (B)
Test II

Name Key

1. Consider the method ($U_i^k = U(x_i, t_k)$)

$$\frac{U_i^{k+1}-U_i^{k-1}}{2dt} = \frac{U_{i+1}^k-(U_i^{k+1}+U_i^{k-1})+U_{i-1}^k}{dx^2}$$

a. Calculate the truncation error, when this is used to approximate $u_t = u_{xx}$. What can you say about the consistency of this method?

$$T = u_{ttt}\frac{dt^2}{6} - u_{xxxx}\frac{dx^2}{12} + u_{tt}\frac{dt^2}{dx^2} + \dots$$

4

Consistent only if $\frac{dt}{dx} \to 0$

b. Use the Fourier method to analyze stability, that is, plug in $U_i^k = a(t_k)e^{Imx_i}$. (Hint: $e^{Imdx} + e^{-Imdx} = 2cos(m * dx)$)

$$(1+r)\lambda^2 - 2r\cos(mdx)\lambda + (r-1) = 0 \qquad r = \frac{2dt}{dx^2}$$

$$\lambda = \frac{r\cos(mdx) \pm \sqrt{1 - r^2\sin^2(mdx)}}{1+r}$$

4

If disc negative:

$$|\lambda_{1,2}|^2 = \frac{r^2\cos^2(mdx) + r^2\sin^2(mdx) - 1}{(1+r)^2} = \frac{r^2-1}{(1+r)^2} = \frac{r-1}{1+r} < 1$$

If disc positive:

$$|\lambda_{1,2}| \le \frac{r+1}{1+r} = 1$$

so unconditionally stable

2

2. Suppose the centered finite difference method:

$$\frac{U_i^{k+1}-2U_i^k+U_i^{k-1}}{dt^2} = c^2\frac{U_{i+1}^k-2U_i^k+U_{i-1}^k}{dx^2}$$

is used to approximate $u_{tt} = c^2 u_{xx}$. This models motion of a wave with speed c, so it is known that the solution at a point (x, t) depends on the initial conditions at points within a distance ct of x, that is, on the initial contitions in the interval $(x - ct, x + ct)$. **Using this information only,** what can you conclude about the relationship between dt and dx required for stability?

If $\frac{dt}{dx} = r$, approx. solution depends on initial data in $(x - \frac{t}{r}, x + \frac{t}{r})$ so must have $\frac{1}{r} \geq C$ or $dt \leq \frac{dx}{c}$

2

3. Consider the problem $u_{xxxx} = u^3 + 1$ in $0 < x < 1$, with boundary conditions $u(0) = u_x(0) = u_x(1) = u(1) = 0$.

   a. Develop a second order finite difference approximation for u_{xxxx} by starting from the approximation:

   $$u_{xxxx}(x_i) \approx \frac{u_{xx}(x_{i+1})-2u_{xx}(x_i)+u_{xx}(x_{i-1})}{dx^2}$$

   and replace each second derivative by a centered difference approximation.

   $$\frac{u_{i+2} - 4u_{i+1} + 6u_i - 4u_{i-1} + u_{i-2}}{dx^4} \cong u_{xxxx}(x_i)$$

   b. Below is a Fortran90 program designed to solve the boundary value problem stated above, with $u_{xxxx}(x_i)$ approximated by the finite difference formula from part (a). Here u(i) is the solution at x(i) = i*h, and we approximate the boundary conditions using $u(0) = 0, u(1) - u(-1) = 0, u(n) = 0, u(n+1) - u(n-1) = 0$.
   The nonlinear relaxation method is used (Newton's method, with the Jacobian replaced by its diagonal, and with a parameter w) in this program. Finish the incomplete statement (Fortran syntax does not have to be correct!)

```
parameter (n=50)
double precision u(-1:n+1)
niter = 10000
w = 1.9
h = 1.d0/n
u(-1:n+1) = 0
do iter=1,niter
   do i=1,n-1
      u(i) = u(i) - w * [ u(i+2) - 4.u(i+1) + 6.u(i)
 -4u(i-1) + u(i-2) - h^4(u(i)^3+1) ] / (6 - h^4 3u(i)^2)
   end do
   u(-1) = u(1)
   u(n+1) = u(n-1)
end do
print *, u
stop
end
```

3

4. a. Find the eigenvalues of $u'' = \lambda u$ with $u'(0) = u'(\pi) = 0$. (Hint: the eigenfunctions are $u_k(x) = cos(kx)$.

1

$u_k'' = -k^2 u_k$ so $\lambda = -k^2$ integer k

b. Approximate this with a centered finite difference problem, and find the eigenvalues of this discrete problem. (Hints: the eigenvectors are $U_k(x_i) = cos(kx_i)$, where $x_i = ih, h = \pi/N$; also, $cos(a + b) = cos(a)cos(b) - sin(a)sin(b)$ and $1 - cos(\theta) = 2sin^2(\theta/2)$.)

2

$$\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} = \lambda u_i \quad \text{so} \quad \lambda = \frac{2\cos(kh) - 2}{h^2} = \frac{-4}{h^2}\sin^2\left(\frac{kh}{2}\right)$$

$$\frac{\cos k(x+h) - 2\cos kx + \cos k(x-h)}{h^2} = \frac{2\cos(kh) - 2}{h^2}\cos(kx)$$

c. Show that for fixed k, as $h \to 0$, the k^{th} eigenvalue of the finite difference problem (b) converges to the k^{th} eigenvalue of the PDE problem (a).

2

$$-\frac{4}{h^2}\sin^2\left(\frac{kh}{2}\right) \to -\frac{4}{h^2}\left(\frac{kh}{2}\right)^2 = -k^2$$

MATH 5343 (c)
Test II

Name Key

1. Consider the method ($U_i^k = U(x_i, t_k)$)

$$\frac{U_i^{k+1} - U_i^k}{dt} = \frac{U_i^k - U_{i-1}^k}{dx}$$

3

a. Calculate the truncation error, when this is used to approximate $u_t = u_x$. What can you say about the consistency of this method?

$$T = \frac{u(x,t+dt) - u(x,t)}{dt} - \frac{u(x,t) - u(x-dx,t)}{dx} = \frac{\cancel{u} + u_t dt + u_{tt}\frac{dt^2}{2} + \ldots - \cancel{u}}{dt}$$

$$- \frac{u - (u - u_x dx + u_{xx}\frac{dx^2}{2} \ldots)}{dx} = u_t + u_{tt}\frac{dt}{2} + \ldots - u_x + u_{xx}\frac{dx}{2}$$

$$= u_{tt}\frac{dt}{2} + u_{xx}\frac{dx}{2} + \ldots$$

consistent

3

b. Use the Fourier method to analyze stability, that is, plug in $U_i^k = a_k e^{Imx_i}, I = \sqrt{-1}$.

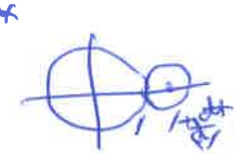
$$\frac{a_{k+1} e^{Imx_i} - a_k e^{Imx_i}}{dt} = \frac{a_k e^{Imx_i} - a_k e^{Im(x_i - dx)}}{dx}$$

$$a_{k+1} - a_k = \frac{dt}{dx} a_k \left(1 - e^{-Imdx}\right)$$

$$\lambda + \left(-1 - \frac{dt}{dx}\left(1 - e^{-Imdx}\right)\right) = 0$$

$$\lambda = 1 + \frac{dt}{dx} - \frac{dt}{dx} e^{-Imdx}$$

always unstable



2

2. Explain why the fact that the exact solution at $(x,t), t > 0$ of the heat equation $u_t = Du_{xx}, u(x,0) = h(x)$ depends on the initial conditions at all points x, shows that the usual explicit finite difference method cannot possibly be stable if dt and dx go to 0 with any constant ratio $dt/dx = r$.

f.d. solution will always depend on initial values in interval $\left(x - \frac{t}{r} \le s \le x + \frac{t}{r}\right)$ so can't be converging to an exact solution which depends on initial values outside this finite interval.

3

3. Analyze the stability of the following approximation to $u_{tt} = c^2 u_{xx}$:

$$\frac{U_i^{k+1} - 2U_i^k + U_i^{k-1}}{dt^2} = c^2 \frac{U_{i+1}^{k+1} - 2U_i^{k+1} + U_{i-1}^{k+1}}{dx^2}$$

Hint: $e^{\theta} - 2 + e^{-\theta} = -4sin^2(\theta/2)$

$$\frac{a_{k+1} - 2a_k + a_{k-1}}{dt^2} e^{Imx_i} = \frac{c^2 a_{k+1}}{dx^2}\left(e^{Im(x_i+dx)} - 2e^{Imx_i} + e^{Im(x_i-dx)}\right)$$

$$a_{k+1} - 2a_k + a_{k-1} = \frac{c^2 dt^2}{dx^2} a_{k+1}\left(-4\sin^2\frac{mdx}{2}\right) \to \lambda^2(1+r) - 2\lambda + 1 = 0$$

$$\lambda = \frac{1 \pm \sqrt{r}\,i}{1+r} \quad |\lambda|^2 = \frac{1}{1+r} < 1 \text{ so always stable} \quad r \geq 0$$

4. a. Calculate the truncation error for the following approximation to $u_{xxxx} = 1$, where $U_i = U(x_i)$.

3

$$\frac{U_{i+2} - 4U_{i+1} + 6U_i - 4U_{i-1} + U_{i-2}}{dx^4} = 1$$

$$1 \quad U_{i+2} = u + u'2h + \frac{u''}{2}(2h)^2 + \frac{u'''}{6}(2h)^3 + \frac{u^{IV}}{24}(2h)^4 + \frac{u^{V}(2h)^5}{120} + \frac{u^{VI}(2h)^6}{720} \ldots$$

$$-4 \quad U_{i+1} = u + u'h + \frac{u''}{2}h^2 + \frac{u'''}{6}h^3 + \frac{u^{IV}}{24}h^4 + \frac{u^{V}h^5}{120} + \frac{u^{VI}h^6}{720} + \ldots$$

$$6 \quad U_i = u$$

$$-4 \quad U_{i-1} = u - u'h + \frac{u''}{2}h^2 - \frac{u'''}{6}h^3 + \frac{u^{IV}h^4}{24} - \frac{u^{V}h^5}{120} + \frac{u^{VI}h^6}{720} + \ldots$$

$$1 \quad U_{i-2} = u - u'2h + \frac{u''(2h)^2}{2} - \frac{u'''}{6}(2h)^3 + \frac{u^{IV}(2h)^4}{24} - \frac{u^{V}(2h)^5}{120} + \frac{u^{VI}(2h)^6}{720} + \ldots$$

odd term all 0

$$\text{numerator} = u(1-4+6-4+1) + \frac{u''}{2}h^2(4-4-4+4) + \frac{u^{IV}}{24}h^4(16-4-4+16)$$

$$+ \frac{u^{VI}h^6}{720}(64-4-4+64)\ldots = u^{IV}h^4 + \frac{1}{6}u^{VI}h^6 + \ldots$$

$$T = \frac{u^{IV}h^4 + \frac{1}{6}u^{VI}h^6 + \ldots}{h^4} - 1 = \frac{1}{6}u^{VI}dx^2 + \ldots$$

3

b. Assume the boundary conditions for (a) are $u(0) = u_{xx}(0) = u(\pi) = u_{xx}(\pi) = 0$. Then (a) is a system of linear equations of the form $Au = f$. Given that the eigenvectors of the matrix A are $W_i = sin(mx_i)$, for integer m, find the eigenvalues. Hint: $sin(m(x+dx)) + sin(m(x-dx)) = 2sin(mx)cos(m\ dx)$ and $cos(2\theta) = 2cos^2(\theta) - 1$.

$$(Aw)_i = \frac{1}{dx^4}\Big[\sin m(x+2dx) - 4\sin m(x+dx) + 6\sin mx + \sin m(x-2dx) - 4\sin m(x-dx)\Big]$$

$$= \frac{1}{dx^4}\Big[2\cos(2m\,dx) - 8\cos(m\,dx) + 6\Big] w_i$$

$$\lambda_m = \frac{1}{dx^4}\Big[2(2\cos^2(m\,dx) - 1) - 8\cos(m\,dx) + 6\Big]$$

$$= \frac{4}{dx^4}(\cos m\,dx - 1)^2 \geq 0$$

c. Write out the Gauss-Seidel iteration to solve the finite difference equations of (a).

2

$$u_i^{k+1} = \left(dx^4 + 4u_{i-1}^{k+1} + 4u_{i+1}^{k} - u_{i-2}^{k+1} - u_{i+2}^{k}\right) / 6$$

d. Is the Gauss-Seidel iteration of part (c) guaranteed to converge? Justify your answer.

1

yes, A is positive definite

10

5. True or False:

T a. Choosing the shift parameter closer to an eigenvalue will generally make the shifted inverse power method converge more rapidly to this eigenvalue.

T b. If SOR is applied to solve the equations of Problem 4a, it is guaranteed to converge for $0 < \omega < 2$.

T c. It is possible to apply the shifted inverse power method to the generalized eigenvalue problem $Az = \lambda Bz$, where A and B are banded matrices, without having to deal with full matrices.

T d. Successive overrelaxation applied to $Ax = b$, with $\omega = 1$, is guaranteed to converge if the matrix A is diagonal dominant or positive definite or negative definite.

F e. For the transport equation $u_t = -\nabla \bullet (u\mathbf{v})$, the boundary conditions should be specified on the part of the boundary where $\mathbf{v} \bullet \mathbf{n}$ is positive.

F f. The explicit upwind approximation to the transport equation is unconditionally stable.

T g. For the diffusion equation $u_t = Du_{xx}$, the speed of diffusion is infinite, while for the wave equation $u_{tt} = c^2 u_{xx}$ the velocity is finite.

T h. If the shifted inverse power method is used to find an eigenvalue of a matrix, with fixed shift, the LU decomposition found on the first iteration can be used on subsequent iterations to decrease the computer time for these iterations.

T i. If an implicit finite difference method is used to solve $U_t = U_{xx} + U_{yy} + U_{zz}$, you will need to solve a large linear system each time step which is banded, but sparse even inside the band.

F j. Nonlinear overrelaxation generally takes more work per iteration than Newton's method, but fewer iterations to converge.