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A second look at the second law

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ABSTRACT

It is commonly argued that the spectacular increase in order which has occurred on Earth does not violate the second law of thermodynamics because the Earth is an open system, and anything can happen in an open system as long as the entropy increases outside the system compensate the entropy decreases inside the system. However, if we define "X-entropy" to be the entropy associated with any diffusing component X (for example, X might be heat), and, since entropy measures disorder, "X-order" to be the negative of X-entropy, a closer look at the equations for entropy change shows that they not only say that the X-order cannot increase in a closed system, but that they also say that in an open system the X-order cannot increase faster than it is imported through the boundary. Thus the equations for entropy change do not support the illogical "compensation" idea; instead, they illustrate the tautology that "if an increase in order is extremely improbable when a system is closed, it is still extremely improbable when the system is open, unless something is entering which makes it *not* extremely improbable". Thus, unless we are willing to argue that the influx of solar energy into the Earth makes the appearance of spaceships, computers and the Internet not extremely improbable, we have to conclude that the second law has in fact been violated here.

1. Compensation

It is probably fair to say that the majority view of science today holds that physics explains all of chemistry, chemistry explains all of biology, and biology completely explains the human mind; thus, physics alone explains the human mind, and all it does.

In fact, since there are only four known forces of physics (the gravitational, electromagnetic and strong and weak nuclear forces), this means that these four forces must explain everything that has happened on Earth, according to this majority view. For example, Peter Urone, in *College Physics* [1], writes "One of the most remarkable simplifications in physics is that only four distinct forces account for all known phenomena".

In my 2000 *Mathematical Intelligencer* article, "A Mathematician's View of Evolution" [2], I argued against this view, asserting that the increase in order which has occurred on Earth seems to violate the second law of thermodynamics in a spectacular way. I wrote:

> I imagine visiting the Earth when it was young and returning now to find highways with automobiles on them, airports with jet airplanes, and tall buildings full of complicated equipment, such as televisions, telephones and computers. Then I imagine the construction of a gigantic computer model which starts with the initial conditions on Earth 4 billion years ago and tries to simulate the effects that the four known forces of physics would have on every atom and every subatomic particle on our planet. If we ran such a simulation out to the present day, would it predict that the basic

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forces of Nature would reorganize the basic particles of Nature into libraries full of encyclopedias, science texts and novels, nuclear power plants, aircraft carriers with supersonic jets parked on deck, and computers connected to laser printers, CRTs and keyboards? If we graphically displayed the positions of the atoms at the end of the simulation, would we find that cars and trucks had formed, or that supercomputers had arisen? Certainly we would not, and I do not believe that adding sunlight to the model would help much.

Anyone who has made such an argument is familiar with the standard reply: the Earth is an open system, it receives energy from the sun, and entropy can decrease in an open system, as long as it is "compensated" somehow by a comparable or greater increase outside the system. For example, Isaac Asimov, in the *Smithsonian* journal [3], recognizes the apparent problem:

> You can argue, of course, that the phenomenon of life may be an exception [to the second law]. Life on earth has steadily grown more complex, more versatile, more elaborate, more orderly, over the billions of years of the planet's existence. From no life at all, living molecules were developed, then living cells, then living conglomerates of cells, worms, vertebrates, mammals, finally Man. And in Man is a three-pound brain which, as far as we know, is the most complex and orderly arrangement of matter in the universe. How could the human brain develop out of the primeval slime? How could that vast increase in order (and therefore that vast decrease in entropy) have taken place?

But Asimov concludes that the second law is not really violated, because

> Remove the sun, and the human brain would not have developed .... And in the billions of years that it took for the human brain to develop, the increase in entropy that took place in the sun was far greater; far, far greater than the decrease that is represented by the evolution required to develop the human brain.

Similarly, Peter Urone, in *College Physics* [1], writes:

> Some people misuse the second law of thermodynamics, stated in terms of entropy, to say that the existence and evolution of life violate the law and thus require divine intervention.... It is true that the evolution of life from inert matter to its present forms represents a large decrease in entropy for living systems. But it is *always* possible for the entropy of one part of the universe to decrease, provided the total change in entropy of the universe increases.

Some other authors appear to feel a little silly suggesting that increases in entropy anywhere in the universe could compensate for decreases on Earth, so they are careful to explain that this "compensation" only works locally; for example in *Order and Chaos* [4], the authors write:

> In a certain sense the development of civilization may appear contradictory to the second law.... Even though society can effect local reductions in entropy, the general and universal trend of entropy increase easily swamps the anomalous but important efforts of civilized man. Each localized, man-made or machine-made entropy decrease is accompanied by a greater increase in entropy of the surroundings, thereby maintaining the required increase in total entropy.

2. The equations of entropy change

Of course the whole idea of compensation, whether by distant or nearby events, makes no sense logically: an extremely improbable event is not rendered less improbable simply by the occurrence of "compensating" events elsewhere. According to this reasoning, the second law does not prevent scrap metal from reorganizing itself into a computer in one room, as long as two computers in the next room are rusting into scrap metal—and the door is open.[1] (Or the thermal entropy in the next room is increasing, though I am not sure how fast it has to increase to compensate computer construction!)

To understand where this argument comes from, we need to look at the equations for entropy change, as given in Appendix D of my 2005 John Wiley book [5], and previously in my 2001 *Mathematical Intelligencer* article [6], "Can ANYTHING Happen in an Open System?".

Consider the diffusion (conduction) of heat in a solid, R, with absolute temperature distribution $U(x, y, z, t)$. The first law of thermodynamics (conservation of energy) requires that

$$Q_t = -\nabla \bullet \mathbf{J}, \tag{1}$$

where Q is the heat energy density ($Q_t = c\rho U_t$) and $\mathbf{J}$ is the heat flux vector. The second law requires that the flux be in a direction in which the temperature is decreasing, i.e.,

$$\mathbf{J} \bullet \nabla U \leq 0. \tag{2}$$

Eq. (2) simply says that heat flows from hot to cold regions—because the laws of probability favor a more uniform distribution of heat energy.

[1] It may be noted that something must actually be entering or leaving a system before it can be considered "open", but if you can see into the next room, electromagnetic radiation at least is entering, and that is what makes the Earth an open system!

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"Thermal entropy" is a quantity that is used to measure randomness in the distribution of heat. The rate of change of thermal entropy, S, is given by the usual definition as

$$S_t = \iiint_R \frac{Q_t}{U} \mathrm{d}V. \tag{3}$$

Using (3) and the first law (1), after doing a (multidimensional) integration by parts, we get

$$S_t = \iiint_R \frac{-\mathbf{J} \bullet \nabla U}{U^2} \mathrm{d}V - \iint_{\partial R} \frac{\mathbf{J} \bullet \mathbf{n}}{U} \mathrm{d}A, \tag{4}$$

where $\mathbf{n}$ is the outward unit normal on the boundary ∂R. From the second law (2), we see that the volume integral is nonnegative, and so

$$S_t \geq -\iint_{\partial R} \frac{\mathbf{J} \bullet \mathbf{n}}{U} \mathrm{d}A. \tag{5}$$

From (5), it follows that $S_t \geq 0$ in an isolated, closed, system, where there is no heat flux through the boundary ($\mathbf{J} \bullet \mathbf{n} = 0$). Hence, in a closed system, the entropy can never decrease. Since thermal entropy measures randomness (disorder) in the distribution of heat, its opposite (negative) can be referred to as "thermal order", and we can say that the thermal order can never increase in a closed system.

Since thermal entropy is quantifiable, the application of the second law to thermal entropy is commonly used as the model problem on which our thinking about the other, less quantifiable, applications is based. The fact that thermal entropy cannot decrease in a closed system, but can decrease in an open system, was used to conclude that, in other applications, any entropy decrease in an open system is possible as long as it is compensated somehow by entropy increases outside this system, so that the total "entropy" (as though there were only one type) in the universe, or any other closed system containing the open system, still increases.

However, there is really nothing special about "thermal" entropy. Heat conduction is just diffusion of heat, and we can define an "X-entropy" (and an X-order $= -X$-entropy), to measure the randomness in the distribution of any other substance X that diffuses; for example, we can let $U(x, y, z, t)$ represent the concentration of carbon diffusing in a solid, and use Eq. (3) again to define this entropy ($c\rho = 1$ now, so $Q_t = U_t$), and repeat the analysis leading to Eq. (5), which now says that the "carbon order" cannot increase in a closed system.[2]

Furthermore, Eq. (5) does not simply say that the X-entropy cannot decrease in a closed system; it also says that, in an open system, the X-entropy cannot decrease faster than it is exported through the boundary, because the boundary integral there represents the rate at which X-entropy is exported across the boundary. To see this, notice that, without the denominator U, the integral in (3) represents the rate of change of total X (energy, if $X =$ heat) in the system; with the denominator it represents the rate of change of X-entropy. Without the denominator U, the boundary integral in (5) represents the rate at which X (energy, if $X =$ heat) is exported through the boundary; with the denominator therefore it must represent the rate at which X-entropy is exported. Although I am certainly not the first to recognize that the boundary integral has this interpretation [see [8], p. 202], this has been noticed by relatively few people, no doubt because usually the special case of isotropic heat conduction or diffusion is assumed, in which case $\mathbf{J} = -K\nabla U$, and then the numerator in the boundary integral is written as $-K\frac{\partial U}{\partial n}$, and in this form it is not obvious that anything is being imported or exported, only that, in a closed system, the boundary integral is zero. Furthermore, entropy as defined by (3) seems to be a rather abstract quantity, and it is hard to visualize what it means to import or export entropy.

Stated in terms of order, Eq. (5) says that the X-order in an open system cannot increase faster than it is imported through the boundary. According to (4), the X-order in a system can decrease in two different ways: it can be converted to disorder (first integral term) or it can be exported through the boundary (boundary integral term). It can increase in only one way: by importation through the boundary.

3. A tautology

The second law of thermodynamics is all about probability; it uses probability at the microscopic level to predict macroscopic change.[3] Carbon distributes itself more and more uniformly in an isolated solid because that is what the laws of probability predict when diffusion alone is operative. Thus the second law predicts that natural (unintelligent) causes will not do macroscopically describable things which are extremely improbable from the microscopic point of view.

2 "Entropy" sounds much more scientific than "order", but note that, in this paper, "order" is simply defined as the opposite of "entropy". Where entropy is quantifiable, such as here, order is equally quantifiable. Physics textbooks also often use the term "entropy" in a less precise sense, to describe the increase in disorder associated with, for example, a plate breaking or a bomb exploding (e.g., [7], p. 651). In such applications, "order" is equally difficult to quantify!

3 In *Classical and Modern Physics*, Kenneth Ford [7] writes "There are a variety of ways in which the second law of thermodynamics can be stated, and we have encountered two of them so far: (1) For an isolated system, the direction of spontaneous change is from an arrangement of lesser probability to an arrangement of greater probability. (2) For an isolated system, the direction of spontaneous change is from order to disorder".

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The reason natural forces can turn a computer or a spaceship into rubble and not vice versa is probability: of all the possible arrangements atoms could take, only a very small percentage could add, subtract, multiply and divide real numbers, or fly astronauts to the moon and back safely.

Of course, we must be careful to define "extremely improbable" events to be events of probability less than some very small threshold: if we define events of probability less than 1% to be extremely improbable, then obviously natural causes *can* do extremely improbable things.[4] But after we define a sufficiently low threshold, everyone seems to agree that "natural forces will rearrange atoms into digital computers" is a macroscopically describable event that is still extremely improbable from the microscopic point of view, and thus forbidden by the second law—at least if this happens in a closed system. But it is not true that the laws of probability only apply to closed systems: if a system is open, you just have to take into account what is crossing the boundary when deciding what is extremely improbable and what is not. What happens in a closed system depends on the initial conditions; what happens in an open system depends on the boundary conditions as well.

The "compensation" counter-argument was produced by people who generalized the model equation for closed systems, but forgot to generalize the equation for open systems. Both equations are only valid for our simple models, where it is assumed that only heat conduction or diffusion is going on; naturally, in more complex situations, the laws of probability do not make such simple predictions. Nevertheless, in "Can ANYTHING Happen in an Open System?" [6], I generalized the equations for open systems to the following tautology, which is valid in all situations:

> *If an increase in order is extremely improbable when a system is closed, it is still extremely improbable when the system is open, unless something is entering which makes it* ***not*** *extremely improbable.*

The fact that order is disappearing in the next room does not make it any easier for computers to appear in our room—unless this order is disappearing *into* our room, and then only if it is a type of order that makes the appearance of computers not extremely improbable, for example, computers. Importing thermal order into an open system may make the temperature distribution less random, and importing carbon order may make the carbon distribution less random, but neither makes the formation of computers more probable.

My conclusion, from "Can ANYTHING Happen in an Open System?" [6] is the following:

> Order can increase in an open system, not because the laws of probability are suspended when the door is open, but simply because order may walk in through the door.… If we found evidence that DNA, auto parts, computer chips, and books entered through the Earth's atmosphere at some time in the past, then perhaps the appearance of humans, cars, computers, and encyclopedias on a previously barren planet could be explained without postulating a violation of the second law here.… But if all we see entering is radiation and meteorite fragments, it seems clear that what is entering through the boundary cannot explain the increase in order observed here.

4. Conclusions

Of course, one can still argue that the spectacular increase in order seen on Earth does not violate the second law because what has happened here is not really extremely improbable. Not many people are willing to make this argument, however; in fact, the claim that the second law does not apply to open systems was invented in an attempt to *avoid* having to make this argument. And perhaps it only seems extremely improbable, but really is not, that, under the right conditions, the influx of stellar energy into a planet could cause atoms to rearrange themselves into nuclear power plants and spaceships and digital computers. But one would think that at least this would be considered an open question, and those who argue that it really *is* extremely improbable, and thus contrary to the basic principle underlying the second law of thermodynamics, would be given a measure of respect, and taken seriously by their colleagues, but we are not.

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[4] If we repeat an experiment 2^k times, and define an event to be "simply describable" (macroscopically describable) if it can be described in m or fewer bits (so that there are 2^m or fewer such events), and "extremely improbable" when it has probability $1/2^n$ or less, then the probability that *any* extremely improbable, simply describable event will *ever* occur is less than $2^{k+m}/2^n$. Thus we just have to make sure to choose n to be much larger than $k + m$. If we flip a billion fair coins, any outcome we get can be said to be extremely improbable, but we only have cause for astonishment if something extremely improbable and simply describable happens, such as "all heads", or "every third coin is tails", or "only every third coin is tails". For practical purposes, almost anything that can be described without resorting to an atom-by-atom (or coin-by-coin) accounting can be considered "macroscopically" describable.

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