Dynamic Systems, Chapter 2

2D dynamical systems

$$\begin{aligned} x_{n+1} &= f(x_n, y_n) \\ y_{n+1} &= g(x_n, y_n) \end{aligned}$$

Fixed point satisfies $x = f(x, y); y = g(x, y)$. Is it stable?

$x_{n+1} = f(x_n, y_n); y_{n+1} = g(x_n, y_n)$ and $x = f(x, y); y = g(x, y)$, so

$$\begin{aligned} x_{n+1} - x &= f(x_n, y_n) - f(x, y) &= \tfrac{\partial f}{\partial x}(x, y)(x_n - x) + \tfrac{\partial f}{\partial y}(x, y)(y_n - y) + ... \\ y_{n+1} - y &= g(x_n, y_n) - g(x, y) &= \tfrac{\partial g}{\partial x}(x, y)(x_n - x) + \tfrac{\partial g}{\partial y}(x, y)(y_n - y) + ... \end{aligned}$$

or

$$e_{n+1} \approx J(x, y)e_n$$

where e_n is the vector $(x_n - x, y_n - y)$ and $J(x, y)$ is the Jacobian matrix:

$$\begin{bmatrix} \frac{\partial f}{\partial x}(x, y) & \frac{\partial f}{\partial y}(x, y) \\ \frac{\partial g}{\partial x}(x, y) & \frac{\partial g}{\partial y}(x, y) \end{bmatrix}$$

Since $e_n \approx J^n(x, y)e_0$, and it is known that the powers of a matrix converge to zero if and only if all its eigenvalues are less than one in absolute value, we see that a fixed point (x, y) is a stable fixed point (sink) if all of the eigenvalues of the Jacobian at (x, y) are less than one in absolute value. If all eigenvalues are greater than one in absolute value, (x, y) is unstable (source); if some are greater than one and some are less, it is called a saddle point, but it still repels nearly all nearby points, so a saddle point is basically unstable also.

Periodic point of period 2 is a point (x_0, y_0) such that $x_1 = f(x_0, y_0); y_1 = g(x_0, y_0)$ and $x_0 = f(x_1, y_1); y_0 = g(x_1, y_1)$, so it is a fixed point of F^2, where $F = (f, g)$, and:

$$\begin{aligned} x_0 &= f(x_1, y_1) &= f(f(x_0, y_0), g(x_0, y_0)) \\ y_0 &= g(x_1, y_1) &= g(f(x_0, y_0), g(x_0, y_0)) \end{aligned}$$

To determine stability for an orbit of period 2, we need to look at the Jacobian of F^2, since both (x_0, y_0) and (x_1, y_1) are fixed points of F^2. Using the chain rule:

$$\begin{bmatrix} \frac{\partial f}{\partial x}(x_1,y_1)\frac{\partial f}{\partial x}(x_0,y_0) + \frac{\partial f}{\partial y}(x_1,y_1)\frac{\partial g}{\partial x}(x_0,y_0) & \frac{\partial f}{\partial x}(x_1,y_1)\frac{\partial f}{\partial y}(x_0,y_0) + \frac{\partial f}{\partial y}(x_1,y_1)\frac{\partial g}{\partial y}(x_0,y_0) \\ \frac{\partial g}{\partial x}(x_1,y_1)\frac{\partial f}{\partial x}(x_0,y_0) + \frac{\partial g}{\partial y}(x_1,y_1)\frac{\partial g}{\partial x}(x_0,y_0) & \frac{\partial g}{\partial x}(x_1,y_1)\frac{\partial f}{\partial y}(x_0,y_0) + \frac{\partial g}{\partial y}(x_1,y_1)\frac{\partial g}{\partial y}(x_0,y_0) \end{bmatrix}$$

=

$$\begin{bmatrix} \frac{\partial f}{\partial x}(x_1,y_1) & \frac{\partial f}{\partial y}(x_1,y_1) \\ \frac{\partial g}{\partial x}(x_1,y_1) & \frac{\partial g}{\partial y}(x_1,y_1) \end{bmatrix} \begin{bmatrix} \frac{\partial f}{\partial x}(x_0,y_0) & \frac{\partial f}{\partial y}(x_0,y_0) \\ \frac{\partial g}{\partial x}(x_0,y_0) & \frac{\partial g}{\partial y}(x_0,y_0) \end{bmatrix}$$

So the Jacobian of F^2 is just the product of the Jacobians of F at the two points in the orbit, and the eigenvalues of this product then determine stability of the orbit.

Example: the Henon map, $f(x, y) = a - x^2 + by; g(x, y) = x$. Note that the Jacobian of F is:

$$\begin{bmatrix} -2x & b \\ 1 & 0 \end{bmatrix}$$

1. Fixed points:

   $$x = f(x, y) = a - x^2 + by$$
   $$y = g(x, y) = x$$

   which implies $x^2+(1-b)x-a = 0$. Provided the discriminant $(1-b)^2+4a$ is positive, there will be two real roots, thus two fixed points, where x is given by the above quadratic formula, and $y = x$. The eigenvalues of the Jacobian are roots of $\lambda^2 + 2x\lambda - b = 0$, if both are less than one in absolute value, the fixed point (x, y) is stable.

2. Period 2 points satisfy:

   $$x = f(f(x, y), g(x, y)) = a - f(x, y)^2 + bg(x, y) = a - (a - x^2 + by)^2 + bx$$
   $$y = g(f(x, y), g(x, y)) = f(x, y) = a - x^2 + by$$

Solving the second equation for $y = (a - x^2)/(1 - b)$ and substituting this into the first equation gives a fourth degree polinomial for x, which is factored:

$$0 = (x^2 - (1-b)x - a + (1-b)^2)(x^2 + (1-b)x - a)$$

The roots of the second quadratic are the fixed points of F, which are also fixed points of F^2; the roots of the first quadratic (if any) are the x values of the (new) period two points. If these roots are called x_0 and x_1, $(y_0 = (a - x_0^2)/(1-b); y_1 = (a - x_1^2)/(1-b))$ then the period two orbit will be stable if the eigenvalues of:

$$\begin{bmatrix} -2x_1 & b \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -2x_0 & b \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 4x_1x_0 + b & -2x_1 b \\ -2x_0 & b \end{bmatrix}$$

are both less than one in absolute value. These eigenvalues are the roots of:

$$\lambda^2 - \lambda[4x_1x_0 + 2b] + b^2 = 0$$

Now let's set $b = 0.4$ and see what happens to the Henon map as a is varied.

Eigenvalues at fixed points $x_0 = -0.3 - \sqrt{0.09 + a}, x_1 = -0.3 + \sqrt{0.09 + a}$

a	x_0	$\lambda_1(x_0)$ $-x_0 - \sqrt{x_0^2 + 0.4}$	$\lambda_2(x_0)$ $-x_0 + \sqrt{x_0^2 + 0.4}$	x_1	$\lambda_1(x_1)$ $-x_1 - \sqrt{x_1^2 + 0.4}$	$\lambda_2(x_1)$ $-x_1 + \sqrt{x_1^2 + 0.4}$
-0.09	-0.300	-0.400	1.000	-0.300	-0.400	1.000
-0.08	-0.400	-0.348	1.148	-0.200	-0.463	0.863
-0.07	-0.441	-0.330	1.213	-0.159	-0.493	0.811
-0.06	-0.473	-0.317	1.263	-0.127	-0.518	0.772
-0.05	-0.500	-0.306	1.306	-0.100	-0.540	0.740
-0.04	-0.524	-0.297	1.345	-0.076	-0.561	0.713
-0.03	-0.545	-0.290	1.380	-0.055	-0.580	0.690
-0.02	-0.565	-0.283	1.412	-0.035	-0.598	0.669
-0.01	-0.583	-0.277	1.443	-0.017	-0.616	0.650
0.00	-0.600	-0.272	1.472	0.000	-0.632	0.632
0.01	-0.616	-0.267	1.499	0.016	-0.649	0.616
0.02	-0.632	-0.262	1.526	0.032	-0.665	0.602
0.03	-0.646	-0.258	1.551	0.046	-0.681	0.588
0.04	-0.661	-0.254	1.575	0.061	-0.696	0.575
0.05	-0.674	-0.250	1.599	0.074	-0.711	0.563
0.06	-0.687	-0.247	1.621	0.087	-0.726	0.551
0.07	-0.700	-0.243	1.643	0.100	-0.740	0.540
0.08	-0.712	-0.240	1.665	0.112	-0.755	0.530
0.09	-0.724	-0.237	1.686	0.124	-0.769	0.520
0.10	-0.736	-0.234	1.706	0.136	-0.783	0.511
0.11	-0.747	-0.232	1.726	0.147	-0.797	0.502
0.12	-0.758	-0.229	1.746	0.158	-0.810	0.494
0.13	-0.769	-0.227	1.765	0.169	-0.824	0.486
0.14	-0.780	-0.224	1.783	0.180	-0.837	0.478
0.15	-0.790	-0.222	1.802	0.190	-0.850	0.470
0.16	-0.800	-0.220	1.820	0.200	-0.863	0.463
0.17	-0.810	-0.218	1.837	0.210	-0.876	0.456
0.18	-0.820	-0.216	1.855	0.220	-0.889	0.450
0.19	-0.829	-0.214	1.872	0.229	-0.902	0.444
0.20	-0.839	-0.212	1.889	0.239	-0.914	0.437
0.21	-0.848	-0.210	1.905	0.248	-0.927	0.432
0.22	-0.857	-0.208	1.922	0.257	-0.939	0.426
0.23	-0.866	-0.206	1.938	0.266	-0.952	0.420
0.24	-0.874	-0.205	1.954	0.274	-0.964	0.415
0.25	-0.883	-0.203	1.969	0.283	-0.976	0.410
0.26	-0.892	-0.202	1.985	0.292	-0.988	0.405
0.27	-0.900	-0.200	2.000	0.300	-1.000	0.400
0.28	-0.908	-0.199	2.015	0.308	-1.012	0.395
0.29	-0.916	-0.197	2.030	0.316	-1.024	0.391
0.30	-0.924	-0.196	2.045	0.324	-1.035	0.386

Eigenvalues at periodic points $x_0 = 0.3 - \sqrt{a - 0.27}, x_1 = 0.3 + \sqrt{a - 0.27}$

a	x_0	x_1	λ_1 $2x_1x_0 + 0.4 - 2\sqrt{x_1^2x_0^2 + 0.4x_1x_0}$	λ_2 $2x_1x_0 + 0.4 + 2\sqrt{x_1^2x_0^2 + 0.4x_1x_0}$
0.27	0.300	0.300	0.160	1.000
0.28	0.200	0.400	0.168	0.952
0.29	0.159	0.441	0.177	0.903
0.30	0.127	0.473	0.188	0.852
0.31	0.100	0.500	0.200	0.800
0.32	0.076	0.524	0.215	0.745
0.33	0.055	0.545	0.233	0.687
0.34	0.035	0.565	0.257	0.623
0.35	0.017	0.583	0.292	0.548
0.36	0.000	0.600	0.400*	0.400*
0.37	-0.016	0.616	0.400*	0.400*
0.38	-0.032	0.632	0.400*	0.400*
0.39	-0.046	0.646	0.400*	0.400*
0.40	-0.061	0.661	0.400*	0.400*
.	.	.	.	.
.	.	.	.	.
0.70	-0.356	0.956	0.400*	0.400*
0.71	-0.363	0.963	0.400*	0.400*
0.72	-0.371	0.971	0.400*	0.400*
0.73	-0.378	0.978	0.400*	0.400*
0.74	-0.386	0.986	0.400*	0.400*
0.75	-0.393	0.993	0.400*	0.400*
0.76	-0.400	1.000	-0.400	-0.400
0.77	-0.407	1.007	-0.548	-0.292
0.78	-0.414	1.014	-0.623	-0.257
0.79	-0.421	1.021	-0.687	-0.233
0.80	-0.428	1.028	-0.745	-0.215
0.81	-0.435	1.035	-0.800	-0.200
0.82	-0.442	1.042	-0.852	-0.188
0.83	-0.448	1.048	-0.903	-0.177
0.84	-0.455	1.055	-0.952	-0.168
0.85	-0.462	1.062	-1.000	-0.160
0.86	-0.468	1.068	-1.047	-0.153
0.87	-0.475	1.075	-1.094	-0.146
0.88	-0.481	1.081	-1.140	-0.140
0.89	-0.487	1.087	-1.185	-0.135
0.90	-0.494	1.094	-1.230	-0.130
0.91	-0.500	1.100	-1.274	-0.126

Thus, to summarize:

	$x_0 = -0.3 - \sqrt{0.09 + a}$	$x_1 = -0.3 + \sqrt{0.09 + a}$	period-2 orbit
$-0.09 < a < 0.27$	unstable	stable	-
$0.27 < a < 0.85$	unstable	unstable	stable