2.3 Logarithmic Functions and Models

Definition – The base *b* logarithm of *x*, $\log_b x$, is the power to which we need to raise *b* in order to get *x*. Symbolically, $\log_b x = y$ means $b^y = x$.

We have two logarithmic bases that are used most frequently and they are the common log, base 10, and the natural log, base e. These two are so widely used that they have their own special notations: $\log_{10} x = \log x$ and $\log_e x = \ln x$.

Examples: Rewrite in the opposite form.

1. $9 = 3^2$

   2 is the power to which I raise 3 to get 9

   $2 = \log_3(9)$

2. $\frac{1}{125} = 5^{-3}$

   $-3 = \log_5\left(\frac{1}{125}\right)$

3. You try it: $2401 = 7^4$

4. $\log_2(8) = 3$

   Base of log is 2, base of exponential is 2

   log equals 3, log is exponent $\therefore$ $2^3 = 8$

5. $\log 10{,}000 = 4$

   No base written, must be $b = 10$

   $10^4 = 10{,}000$

6. You try it: $\ln 7 = x$

Because these two are so commonly used, they are on our calculators. It is impossible for a calculator to have a shortcut button for every logarithmic base so we have a change of base formula that allows us to enter any base on the calculator.

Change of Base Formula:

$$\log_b a = \frac{\log a}{\log b} = \frac{\ln a}{\ln b}$$

Examples: Use logarithms to solve.

1. $4^x = 3$

① rewrite

$x = \log_4(3)$

② Evaluate $x = \frac{\log 3}{\log 4} \approx 0.7925$

2. $6^{3x+1} = 30$

① Rewrite

$3x+1 = \log_6(30)$

② Solve: $3x = \log_6(30) - 1$

$x = \frac{\log_6(30) - 1}{3}$

③ Evaluate:

$x = \frac{\frac{\log(30)}{\log(6)} - 1}{3} = 0.2994$

3. $5.3(10^x) = 2$

① Isolate the exponential part

$10^x = \frac{2}{5.3}$

② Rewrite

$x = \log\left(\frac{2}{5.3}\right)$

③ Evaluate

$x = -0.4232$

4. $4(1.5)^{2x-1}=8$

① Isolate: $(1.5)^{2x-1}=2$

② Rewrite: $\log_{1.5}(2)=2x-1$

③ Solve: $\log_{1.5}(2)+1=2x$

$\frac{\log_{1.5}(2)+1}{2}=x$

④ Evaluate:

$x=\frac{\frac{\log(2)}{\log(1.5)}+1}{2}$

$x\approx 1.3548$

Definition – A logarithmic function has the form $f(x)=\log_b x+C$ or, alternatively, $f(x)=A\ln x+C$

Logarithmic Identities – The following identities hold for all positive bases $a\neq 1$ and $b\neq 1$, all positive numbers *x* and *y*, and for every real number *r*. The identities follow from the laws of exponents.

1. $\log_b(xy)=\log_b x+\log_b y$
2. $\log_b\left(\frac{x}{y}\right)=\log_b x-\log_b y$
3. $\log_b x^r=r\log_b x$
4. $\log_b b=1;\ \log_b 1=0$
5. $\log_b\left(\frac{1}{x}\right)=-\log_b x$
6. $\log_b x=\frac{\log_a x}{\log_a b}$

Example: How long will it take a $500 investment to be worth $700 if it is continuously compounded at 10% per year?

$700=500e^{.10t}$

① Isolate

$\frac{700}{500}=e^{.10t}$

$\frac{7}{5}=e^{.10t}$

② Rewrite

$.10t=\ln\left(\frac{7}{5}\right)$

③ Solve

$t=\frac{\ln\left(\frac{7}{5}\right)}{.10}\approx 3.4$

It will take about 3.4 yrs.

Example: How long, to the nearest year, will it take an investment to triple if it is continuously compounded at 12% per year?

Assume $P = 100$, then $A = 300$

$300 = 100e^{.12t}$

$3 = e^{.12t}$

$.12t = \ln(3)$

$t = \frac{\ln(3)}{.12} \approx 9.155$

It will take about 9 yrs to triple.

Definition – An exponential decay function has the form $Q(t) = Q_0 e^{-kt}$. Q_0 represents the value of *Q* at time *t = 0*, and *k* is the decay constant. The decay constant *k* and half-life t_h for *Q* are related by $k \cdot t_h = \ln 2$.

Definition – An exponential growth function has the form $Q(t) = Q_0 e^{kt}$. Q_0 represents the value of *Q* at time *t = 0*, and *k* is the growth constant. The growth constant *k* and doubling time t_d for *Q* are related by $k \cdot t_d = \ln 2$.

Examples: Find the associated exponential growth or decay function.

1. *Q = 1000* when *t = 0*; half-life = 3

$Q_0 = 1000$

$t_h = 3$

$K(3) = \ln 2$

$K = \frac{\ln(2)}{3}$

$Q(t) = 1000e^{-\frac{\ln(2)}{3}t}$

2. *Q = 2000* when *t = 0*; doubling time = 2

$Q_0 = 2000$

$t_d = 2$

$K(2) = \ln 2$

$K = \frac{\ln(2)}{2}$

$Q(t) = 2000e^{\frac{\ln(2)}{2}t}$

Examples: Convert the given exponential function to the form indicated. Round all coefficients to four significant digits.

1. $f(x) = 4e^{2x}$ to the form $f(x) = A(b)^x$

$A = 4$

$b = e^2 \approx 7.389$

$f(x) = 4(7.389)^x$

2. $f(t) = 2.1(1.001)^t$ to the form $Q(t) = Q_0 e^{kt}$

$Q_0 = 2.1$

$e^k = 1.001$

$k = \ln(1.001) \approx 0.0009995$

$Q(t) = 2.1 e^{0.0009995t}$

3. You try it: $f(t) = 10(0.987)^t$ to the form $Q(t) = Q_0 e^{-kt}$

Example: Soon after taking an aspirin, a patient has absorbed 300 mg of the drug. If the amount of aspirin in the bloodstream decays exponentially, with half being removed every 2 hours, find, to the nearest 0.1 hours, the time it will take for the amount of aspirin in the bloodstream to decrease to 100 mg.

find t

Q

absorbed 300 mg $\Rightarrow Q_0 = 300$

Decays exponentially $\rightarrow Q(t) = Q_0 e^{-kt}$

half removed (half-life) 2 hours $t_h = 2$

$k = \frac{\ln(2)}{2} \approx 0.3466$

Equation: $Q(t) = 300e^{-0.3466t}$ ← how do I know it is negative?

$100 = 300e^{-0.3466t}$

$\frac{1}{3} = e^{-0.3466t}$

$-0.3466t = \ln(1/3)$

$t = \frac{\ln(1/3)}{-0.3466} \approx 3.17$

$\boxed{\text{About } 3.2 \text{ hrs}}$

Example: After a large number of drinks, a person has a BAC of 200 mg/dL. If the amount of alcohol in the blood decays exponentially, with one fourth being removed every hour, find the time it will take for the person's BAC to decrease to 80 mg/dL

Initial $= Q_0 = 200$

Decays exponentially $\Rightarrow Q(t) = Q_0 e^{-Kt}$

$\frac{1}{4}$ removed is not a half life but it will give us some information. $\frac{1}{4}$ of 200 is 50 mg/dL removed in first hour. This gives a point $(1, 150)$ that can be used to find K.

$$150 = 200e^{-K(1)}$$

$$\frac{150}{200} = e^{-K}$$

$$-K = \ln\left(\frac{150}{200}\right)$$

$$-K \approx -0.2877$$

We can now use this in our model as $-K$ to find t when $Q = 80$.

$$80 = 200e^{-0.2877t}$$

$$\frac{80}{200} = e^{-0.2877t}$$

$$-0.2877t = \ln\left(\frac{80}{200}\right)$$

$$t = \frac{\ln(80/200)}{-0.2877} \approx 3.18$$ or approximately 3.2 hours.

*Note: At 80 mg/dL you can be arrested for drunk driving.