3.3 Annuities, Loans and Bonds

Formula – A sinking fund is an account earning compound interest into which you make periodic deposits. Suppose that an account has an annual percentage rate of *r* compounded *m* times per year so that $i = r/m$ is the interest rate per compounding period. If you make a payment of PMT at the end of each period, then the future value after *t* years, or $n = mt$ periods will be

$$FV = PMT\frac{(1+i)^n - 1}{i}$$

We can use the same formula if given the FV and wanting to find the PMT by using division in the last step rather than multiplication.

Examples: Find the FV of the sinking fund.

1. $100 is deposited monthly for 10 years at 5%

$i = \frac{.05}{12}$

$n = 12(10) = 120$

$$FV = 100\frac{(1+.05/12)^{120} - 1}{\left(\frac{.05}{12}\right)} = 100\,\frac{0.64700949}{0.0041666..} = \$15{,}528.23$$

2. $50 is deposited monthly for 25 years at 3%

$i = \frac{.03}{12} = 0.0025$

$n = 12(25) = 300$

$$FV = 50\,\frac{(1+.0025)^{300} - 1}{.0025} = 50\,\frac{1.115019557}{.0025} = \$22{,}300.39$$

Examples: Find the PMT for the sinking fund.

1. We want $20,000 in a fund paying 4% per year, with monthly payments for 10 years.

$i = \frac{.04}{12}$

$n = 12(10) = 120$

$$20{,}000 = PMT\,\frac{(1+.04/12)^{120} - 1}{\left(\frac{.04}{12}\right)} \rightarrow 20{,}000 = PMT(147.249804724)$$

$$\$135.82 = PMT$$

2. We want $100,000 in a fund paying 7% per year, with quarterly payments for 20 years.

$i = \frac{.07}{4} = 0.0175$

$n = 4(20) = 80$

$$100{,}000 = PMT\,\frac{(1+0.0175)^{80} - 1}{0.0175}$$

$$100{,}000 = PMT(171.7934243)$$

$$PMT = \$582.09$$

Formula – An annuity is an account earning compound interest from which periodic withdrawals are made. Suppose that the account has an annual rate of *r* compounded *m* times per year, so that $i = r/m$ is the interest rate per compounding period. Suppose also that the account starts with a balance of PV. If you receive a payment of PMT at the end of each compounding period, and the account is down to $0 after *t* years, or $n = mt$ periods, then

$$PV = PMT\frac{1-(1+i)^{-n}}{i}$$

Notice we can use this same formula to also solve for PMT. $\longrightarrow$ $\frac{PV \cdot i}{1-(1+i)^{-n}} = PMT$

Examples: Find the periodic withdrawals for the annuities given.

1. $100,000 at 3%, paid out monthly for 20 years.

$i = \frac{.03}{12} = 0.0025$

$n = 12(20) = 240$

$PMT = \frac{100{,}000(0.0025)}{(1-(1+0.0025)^{-240})} = \$554.60$

2. $25 million at 2%, paid out monthly for 20 years.

$i = \frac{.02}{12}$

$n = 12(20) = 240$

$PMT = \frac{25{,}000{,}000\left(\frac{.02}{12}\right)}{\left(1-\left(1+\frac{.02}{12}\right)^{-240}\right)} = \$126{,}470.83$

Fact: From a lenders point of view, a loan is an annuity. Therefore we use the same formula.

Examples: Find the periodic payments on the loans given.

1. $10,000 borrowed at 9% for 4 years, with monthly payments.

$i = \frac{.09}{12} = 0.0075$

$n = 12(4) = 48$

$PMT = \frac{10{,}000(0.0075)}{(1-(1+.0075)^{-48})} = \$248.85$

2. $150,000 borrowed at 5.6% for 30 years, with monthly payments.

$i = \frac{0.056}{12}$

$n = 12(30) = 360$

$PMT = \frac{150{,}000\left(\frac{0.056}{12}\right)}{\left(1-\left(1+\frac{.056}{12}\right)^{-360}\right)} = \$861.12$

Example – Your pension plan is an annuity with a guaranteed return of 3% per year (compounded monthly). You would like to retire with a pension of \$5000 per month for 20 years. If you work 40 years before retiring, how much must you and your employer deposit each month into the fund?

This problem has two parts. If we want \$5000/month for 20 yrs we need to determine the PV for the annuity. Next, this amount becomes the FV of the sinking fund.

$i = \frac{.03}{12} = 0.0025$

$$PV = 5000\frac{\left(1-(1+.0025)^{-240}\right)}{.0025} = 901{,}554.57$$

$$901{,}554.57 = PMT\,\frac{(1.0025)^{480}-1}{.0025}$$

$$901{,}554.57 = PMT(926.0595011 88)$$

$\rightarrow$ $PMT = 973.54$

They need to deposit \$973.54 each month for 40 yrs.

Example – While shopping for a car loan, you get the following offers: SSL is willing to loan you \$10,000 at 9% interest for 4 years. FFB&T will loan you the \$10,000 at 7% for 3 years. Both require monthly payments. You can afford to pay only \$250 per month. Which loan, if either, can you take?

SSL $\quad i = \frac{.09}{12} = 0.0075 \quad n = 12(4) = 48$

$$PMT = \frac{10{,}000(.0075)}{\left(1-(1+0.0075)^{-48}\right)}$$

$PMT = 248.85$ $\leftarrow$ take this one

FFB&T $\quad i = \frac{.07}{12} \quad n = 3(12) = 36$

$$PMT = \frac{10{,}000\left(\frac{.07}{12}\right)}{\left(1-\left(1+\frac{.07}{12}\right)^{-36}\right)}$$

$PMT = 308.77$

Example – Suppose you inherit \$10,000 when you are 35 years old and you use that to start your retirement account. You then deposit \$150 per month for 20 years at 3% per year with monthly compounding. How much would you have in total at the end of those 20 years?

The \$10,000 is a lump sum that earns compound interest:

$$FV = 10{,}000\left(1+\frac{.03}{12}\right)^{240} = 18{,}207.55$$

The monthly deposit is a sinking fund:

$$FV = 150\,\frac{\left(1+\frac{.03}{12}\right)^{240}-1}{\left(\frac{.03}{12}\right)} = 49{,}245.30$$

Total is $18{,}207.55 + 49{,}245.30 = \$67{,}452.85$