4.2 Using Matrices to Solve Systems of Equations

Definition – A linear equation in the *n* variables $x_1, x_2, ..., x_n$ has the form

$$a_1x_1 + a_2x_2 + ... + a_nx_n = b$$

The numbers a_i are called the coefficients, and the number *b* is called the constant term, or right-hand side.

Definition – A matrix is a rectangular array of numbers. The augmented matrix of a system of linear equations is the matrix whose rows are the coefficients rows of the equations.

Example: The augmented matrix of the system $\begin{cases} x+y=3 \\ x-y=1 \end{cases}$ is $\left[\begin{array}{cc:c} 1 & 1 & 3 \\ 1 & -1 & 1 \end{array}\right]$.

This allows us to take the variables out of the process. We will always have our equations line up appropriately, so each column represents a specific variable. Just as we had an algebraic way of solving a system, we have a way of solving a matrix.

Elementary Row Operations:

1. We can multiply or divide an entire row by any nonzero number.
2. We can multiply a row by a nonzero number and add or subtract a multiple of another row.
3. We can switch the order of the rows.

Fact – Gauss-Jordan row reduction uses these operations to arrive at a matrix in reduced-row echelon form where we have 1's on the diagonals, 0's elsewhere and constants in the far right column.

If you would like to learn more about matrices, you should take a class in matrix algebra. Here at UTEP that would be Math 3323. Instead of using matrices, we will use the same idea and take an organized approach to solving these "harder" systems.

Examples: Use G-J row reduction to solve the given systems.

1. $\begin{cases} x+y=4 \\ x-y=2 \end{cases}$ eliminate x in second row:

$$\begin{array}{r} x+y=4 \\ -x+y=-2 \\ \hline 2y=2 \\ y=1 \end{array}$$

multiply eq2 by -1

Now that $y=1$, back-substitute to find x: $x+1=4$

$x=3$

answer $(x,y) = \boxed{(3,1)}$

2. $\begin{cases} 2x+3y=1 \\ -x-\dfrac{3y}{2}=-\dfrac{1}{2} \end{cases}$ $\left(-x-\dfrac{3y}{2}=-\dfrac{1}{2}\right)2$

$$\begin{aligned} 2x+3y&=1 \\ -2x-3y&=-1 \\ \hline 0&=0 \end{aligned}$$ always true

$$\begin{aligned} 2x+3y&=1 \\ 3y&=1-2x \\ y&=\frac{1-2x}{3} \end{aligned}$$

answer $(x,y)=\boxed{\left(x, \frac{1-2x}{3}\right)}$

3. $\begin{cases} -x+2y-z=0 \\ -x-y+2z=0 \\ 2x \quad\quad -z=4 \end{cases}$ use equation 1 to eliminate all other x's

multiply eq1 by -1

1+2: $$\begin{aligned} x-2y+z&=0 \\ -x-y+2z&=0 \\ \hline -3y+3z&=0 \end{aligned}$$

1+3: $$\begin{aligned} -2x+4y-2z&=0 \\ 2x \quad\quad -z&=4 \\ \hline 4y-3z&=4 \end{aligned}$$ multiply eq1 by 2

Now you have two equations with two variable. Eliminate y from one equation

$$4(-3y+3z=0) \rightarrow -12y+12z=0$$
$$3(4y-3z=4) \rightarrow 12y-9z=12$$
$$\begin{aligned} \hline 3z&=12 \\ z&=4 \end{aligned}$$

Back-substitute with z=4 to get $$\begin{aligned} -3y+3(4)&=0 \\ -3y+12&=0 \\ -3y&=-12 \\ y&=4 \end{aligned}$$

Back substitute with z=4, y=4 to get

$$-x+2(4)-(4)=0$$
$$-x+4=0 \rightarrow x=4$$

answer $(x,y,z)=\boxed{(4,4,4)}$

4. $\left(\begin{cases} x - \frac{1}{2}y = 0 \\ \frac{1}{2}x - \frac{1}{2}z = -1 \\ 3x - y - z = -2 \end{cases}\right.$ $\begin{matrix})\,2 \\ )\,2 \\ \ \end{matrix}$

$$\begin{cases} 2x - y = 0 \\ x - z = -2 \\ 3x - y - z = -2 \end{cases}$$

new equivalent system without fractions.

1+2: mult eq2 by −2

$$\begin{array}{r} 2x - y \qquad\quad = 0 \\ -2x \qquad + 2z = 4 \\ \hline -y + 2z = 4 \end{array}$$

1+3: multiply eq1 by 3 and eq3 by −2

$$\begin{array}{r} 6x - 3y \qquad\quad = 0 \\ -6x + 2y + 2z = 4 \\ \hline -y + 2z = 4 \end{array}$$

they are the same!

Not helpful until you realize this makes the system dependent.

Let x be x then find y and z. Look at equations 1+2 of no fractions

$y = 2x$ and $z = x + 2$

answer: $(x, y, z) = \boxed{(x, 2x, x+2)}$