1.2 Finding Limits Graphically and Numerically

Graphing functions seems fairly straightforward for functions that have a domain of all real numbers. We can choose a few domain points, find the range values that go with them, then plot and join with a smooth curve. However, when the domain has exclusions we need to determine what is going on at or near these values. In order to do this we use what is called a limit.

Informal definition: If *f(x)* becomes arbitrarily close to a single number *L* as *x* approaches *c* from either side, the limit of *f(x)* as *x* approaches *c*, is *L*. The limit is written as

$$\lim_{x\to c} f(x) = L$$

Example: Use a table of values to estimate the limit numerically.

1. $\lim_{x\to 2} \frac{x-2}{x^2-4}$

x	1.9	1.99	2	2.01	2.1
$\frac{x-2}{x^2-4}$	.2564	.2506	?	.2494	.2439

replace x with each value to fill in the table. Now make an educated guess that

$\lim_{x\to 2} \frac{x-2}{x^2-4} = .25$ (or $\frac{1}{4}$)

2. $\lim_{x\to -5} \frac{\sqrt{4-x}-3}{x+5}$

x	-5.1	-5.01	-5	-4.99	-4.9
$f(x)$	-.1662	-.1666	?	-.1667	-.1671

It looks like $f(x) \to -.1666\ldots$ as $x \to -5$, but what is $-.1\bar{6}$? This is a decimal approx for $-\frac{1}{6}$

so $\lim_{x\to -5} \frac{\sqrt{4-x}-3}{x+5} = -\frac{1}{6}$

3. $\lim_{x\to 4} \frac{[x/(x+1)]-(4/5)}{x-4}$ $= .04$ $\left(= \frac{1}{25}\right)$

x	3.9	3.99	4	4.01	4.1
$f(x)$	.0408	.0401	?	.0399	.0392

4. $\lim_{x\to 0} \frac{\sin 4x}{x}$

x	-0.1	-0.01	0	0.01	0.1
$\frac{\sin(4x)}{x}$	.0698	.0698	?	.0698	.0698

this is degree mode, oops.
We will always use radians in calculus.

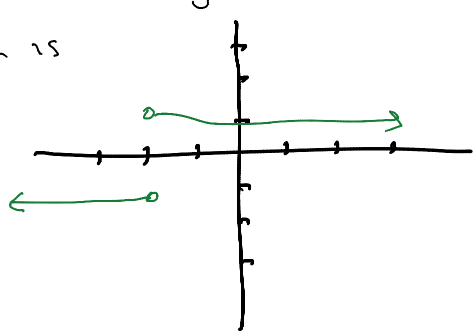
x	-0.1	-0.01	0	0.01	0.1
$\frac{\sin(4x)}{x}$	3.8942	3.9989	?	3.9989	3.8942

this looks better as we can see that $\lim_{x\to 0} \frac{\sin(4x)}{x} = 4$

Examples: Use the graph to find the limit.

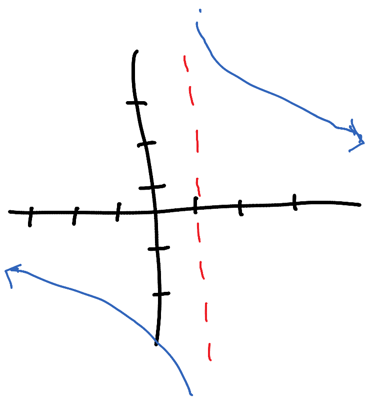
1. $\lim_{x\to -2} \frac{|x+2|}{x+2}$

Notice that the numerator and denominator are the same except in sign for x less than -2. This means the graph is



As $x \to -2$ from the right, the output is 1. However, as we approach from the left $f(x) \to -1$. These values do not match so $\lim_{x\to -2} \frac{|x+2|}{x+2}$ DOES NOT EXIST

2. $\lim_{x\to 1}\frac{5}{x-1}$ Remembering the skills learned in pre-calculus, we know the graph is essentially:



As $x \to 1$, the graph seems to "blow up" to both + and − infinity. This also means that $\lim_{x\to 1}\frac{5}{x-1}$ DOES NOT EXIST

Common Types of Behavior Associated with Nonexistence of a Limit –

1. The function approaches a different number from the right side than it approaches from the left side.

2. The function increases or decreases without bound as *x* approaches *c*. (Goes to infinity)

3. The function oscillates between two fixed values. (Frequently with trigonometric functions)

Formal Definition: Let *f* be a function defined on an open interval containing *c* (except possibly at *c*) and let *L* be a real number. The statement $\lim_{x\to c} f(x) = L$ means that for each ε > 0 there exists a δ > 0 such that if $0 < |x-c| < \delta$, then $|f(x)-L| < \varepsilon$.

Examples: Find the limit *L*. Then find δ > 0 such that $|f(x)-L| < 0.01$ whenever $0 < |x-c| < \delta$.

1. $\lim_{x\to 2}(3x+2) = 8$ using either a table of values or the graph

Start: $|f(x) - L| < 0.01$

is $|3x+2-8| < 0.01$

or $|3x-6| < 0.01$

Factor out the 3: $3|x-2| < 0.01$

divide $|x-2| < \frac{0.01}{3}$

Goal: $|x-2| < \delta$

c

It must be that $\delta = \frac{0.01}{3}$

2. $\lim_{x \to 5}(x^2+4) = 29$

$|f(x)-L| < 0.01$

$|x^2+4-29| < 0.01$

$|x^2-25| < 0.01$

$|x+5||x-5| < 0.01$

our goal is to isolate $|x-5|$. $\{|x-c| < \delta\}$ we can divide by $|x+5|$ as long as it is not zero. Near $x=5$, this is not zero and it is at most 11. {Why 11? Near 5 is 6, and $|6+5| = 11$.}

$|x-5| < \frac{0.01}{|x+5|} \Rightarrow |x-5| < \frac{0.01}{11}$ $\delta = \frac{0.01}{11}$

Examples: Find the limit *L*. Then use the ε-δ definition to prove that the limit is *L*.

1. $\lim_{x \to -3}(2x+5) = -1$ (table or graph)

$|2x+5-(-1)| < \varepsilon$

$|2x+6| < \varepsilon$

$2|x+3| < \varepsilon$

$|x+3| < \frac{\varepsilon}{2}$

We can get $f(x) = 2x+5$ as close to -1 as we want, ε, as long as x is within $\frac{\varepsilon}{2} = \delta$ of -3.

2. $\lim_{x \to 4}\sqrt{x} = 2$

$|f(x)-L| < \varepsilon$

$|\sqrt{x}-2| < \varepsilon$

$|\sqrt{x}+2||\sqrt{x}-2| < \varepsilon|\sqrt{x}+2|$

$|x-4| < \varepsilon|\sqrt{x}+2|$

Multiply by the conjugate of $\sqrt{x}-2$ in order to get to our goal of $|x-c| = |x-4| < \delta$. Now, near 4, $|\sqrt{x}+2|$ is close to $|\sqrt{4}+2| = 4$.

so $|x-4| < \varepsilon(4)$ or $\delta = 4\varepsilon$

3. $\lim_{x \to 6}|x-6| = 0$

This one is boring..

$|f(x)-L| < \varepsilon$

$||x-6|-0| < \varepsilon$

$||x-6|| < \varepsilon$

$|x-6| < \varepsilon$

so $\delta = \varepsilon$.