

1.3 Evaluating Limits Analytically

Properties of Limits (1) – Let b and c be real numbers and let n be a positive integer.

$$1. \lim_{x \rightarrow c} b = b \qquad 2. \lim_{x \rightarrow c} x = c \qquad 3. \lim_{x \rightarrow c} x^n = c^n$$

Properties of Limits (2) – Let b and c be real numbers, let n be a positive integer, and let f and g be functions with the following limits.

$$\lim_{x \rightarrow c} f(x) = L \quad \text{and} \quad \lim_{x \rightarrow c} g(x) = K$$

1. Scalar Multiple: $\lim_{x \rightarrow c} [bf(x)] = bL$
2. Sum of difference: $\lim_{x \rightarrow c} [f(x) \pm g(x)] = L \pm K$
3. Product: $\lim_{x \rightarrow c} [f(x)g(x)] = LK$
4. Quotient: $\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \frac{L}{K}$, provided $K \neq 0$
5. Power: $\lim_{x \rightarrow c} [f(x)]^n = L^n$

These eight limits lead us to some great results that make our job of evaluating limits easier so that we do not have to rely on the time consuming method of tables or even of knowing what the graph looks like.

Limits of Polynomial and Rational Functions –

If p is a polynomial function and c is a real number, then $\lim_{x \rightarrow c} p(x) = p(c)$.

If r is a rational function and given by $r(x) = p(x)/q(x)$ and c is a real number such that

$$q(c) \neq 0, \text{ then } \lim_{x \rightarrow c} r(x) = r(c) = \frac{p(c)}{q(c)}.$$

That is, to evaluate the limit of a polynomial or rational function, evaluate it (if possible).

Limit of a Function Involving a Radical –

Let n be a positive integer. The following limit is valid for all c if n is odd and is valid for $c > 0$ if n is even. $\lim_{x \rightarrow c} \sqrt[n]{x} = \sqrt[n]{c}$

Once again, just evaluate the radical function if the domain allows in order to find the limit.

Limit of a Composite Function –

If f and g are functions such that $\lim_{x \rightarrow c} g(x) = L$ and $\lim_{x \rightarrow c} f(x) = f(L)$, then

$$\lim_{x \rightarrow c} f(g(x)) = f(\lim_{x \rightarrow c} g(x)) = f(L)$$

Limits of Trigonometric Functions – Let c be a real number in the domain of the given trigonometric function.

1. $\lim_{x \rightarrow c} \sin x = \sin c$

2. $\lim_{x \rightarrow c} \cos x = \cos c$

3. $\lim_{x \rightarrow c} \tan x = \tan c$

4. $\lim_{x \rightarrow c} \cot x = \cot c$

5. $\lim_{x \rightarrow c} \sec x = \sec c$

6. $\lim_{x \rightarrow c} \csc x = \csc c$

With all these properties it would be easy to feel overwhelmed. In order to overcome this feeling, we're going to organize our information. The first thing that you need to know is which limits can be found simply by evaluating the function. **In general, as long as c is in the domain of the function, you can evaluate the function to find the limit.** If c is not in the domain of the function we must have some strategies in order to find the limit. First, a fact:

Theorem 1.7 – Let c be a real number and let $f(x) = g(x)$ for all $x \neq c$ in an open interval containing c . If the limit of $g(x)$ as x approaches c exists, then the limit of $f(x)$ also exists and

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} g(x)$$

A Strategy for Finding Limits -

1. Learn to recognize which limits can be evaluated by direct substitution.
2. If the limit of $f(x)$ as x approaches c cannot be evaluated by direct substitution, try to find a function g that agrees with f for all x other than $x = c$. Common techniques involve dividing out and rationalizing.
3. Apply Theorem 1.7 to conclude analytically that $\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} g(x) = g(c)$.
4. Use a graph or table to reinforce your conclusion.

Examples: Find the limits

1. $\lim_{x \rightarrow 1} (-x^2 + 1) = -(1)^2 + 1 = -1 + 1 = 0$ *evaluate a polynomial to find the limit.*
2. $\lim_{x \rightarrow 4} \sqrt[3]{x+4} = \sqrt[3]{4+4} = \sqrt[3]{8} = 2$ *evaluate any radical with odd index*
3. $\lim_{x \rightarrow -3} \frac{2}{x+2} = \frac{2}{-3+2} = \frac{2}{-1} = -2$ *evaluate since -3 is in the domain of the rational function*
4. $\lim_{x \rightarrow \pi} \tan x = \tan \pi = 0$
5. $\lim_{x \rightarrow 5\pi/3} \cos x = \cos \frac{5\pi}{3} = \frac{1}{2}$
6. $\lim_{x \rightarrow -1} \frac{2x^2 - x - 3}{x+1} = \lim_{x \rightarrow -1} \frac{(2x-3)\cancel{(x+1)}}{\cancel{x+1}} = \lim_{x \rightarrow -1} (2x-3) = 2(-1) - 3 = -5$
*-1 is not in the domain
 so we use theorem 1.7*
7. $\lim_{x \rightarrow 2} \frac{x^3 - 8}{x-2} = \lim_{x \rightarrow 2} \frac{\cancel{(x-2)}(x^2 + 2x + 4)}{\cancel{x-2}} = \lim_{x \rightarrow 2} (x^2 + 2x + 4) = (2)^2 + 2(2) + 4 = 12$

anytime evaluating gives $\frac{0}{0}$, you must try something else

$$8. \lim_{x \rightarrow 0} \frac{3x}{x^2 + 2x} = \lim_{x \rightarrow 0} \frac{\cancel{3x}}{\cancel{x}(x+2)} = \lim_{x \rightarrow 0} \frac{3}{x+2} = \frac{3}{0+2} = \frac{3}{2}$$

$$9. \lim_{x \rightarrow 3} \frac{\sqrt{x+1}-2}{x-3} = \lim_{x \rightarrow 3} \frac{1}{\sqrt{x+1}+2} = \frac{1}{\sqrt{3+1}+2} = \frac{1}{\sqrt{4}+2} = \frac{1}{2+2} = \frac{1}{4}$$

factoring won't help so we rationalize the numerators:

$$\frac{\sqrt{x+1}-2}{x-3} \cdot \frac{\sqrt{x+1}+2}{\sqrt{x+1}+2} = \frac{x+1-4}{(x-3)(\sqrt{x+1}+2)} = \frac{\cancel{x-3}}{(\cancel{x-3})(\sqrt{x+1}+2)} = \frac{1}{\sqrt{x+1}+2}$$

Even though these methods work frequently, they do not work for every function. This leads to another theorem:

Theorem 1.8 The Squeeze Theorem – If $h(x) \leq f(x) \leq g(x)$ for all x in an open interval containing c , except possibly at c itself, and if $\lim_{x \rightarrow c} h(x) = L = \lim_{x \rightarrow c} g(x)$ then $\lim_{x \rightarrow c} f(x)$ exists and is equal to L .

Using the Squeeze Theorem, also called the Sandwich Theorem, we can define two special trigonometric limits.

$$*1. \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \quad \text{and} \quad *2. \lim_{x \rightarrow 0} \frac{1 - \cos x}{x} = 0$$

* These will come back in later chapters, memorize them now or write on a notecard for future reference!

Examples: Determine the limit of the trig function, if it exists.

$$1. \lim_{x \rightarrow 0} \frac{3(1 - \cos x)}{x} = 3 \left(\lim_{x \rightarrow 0} \frac{1 - \cos x}{x} \right) = 3(0) = 0$$

$$2. \lim_{\theta \rightarrow 0} \frac{\cos \theta \tan \theta}{\theta} = \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$$

$$\frac{\cos \theta \tan \theta}{\theta} = \frac{\cancel{\cos \theta} \cdot \frac{\sin \theta}{\cancel{\cos \theta}}}{\theta} = \frac{\sin \theta}{\theta}$$

$$3. \lim_{\phi \rightarrow \pi} \phi \sec \phi = \lim_{\phi \rightarrow \pi} \frac{\phi}{\cos \phi} = \frac{\pi}{\cos \pi} = \frac{\pi}{-1} = -\pi$$

$$\phi \sec \phi = \frac{\phi}{\cos \phi}$$