

## 1.4 Continuity and One-sided Limits

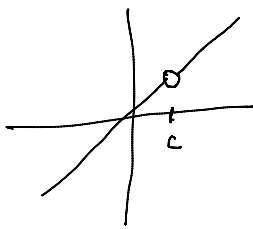
Definition of Continuity – A function  $f$  is continuous at  $c$  if the following three conditions are met.

1.  $f(c)$  is defined.
2.  $\lim_{x \rightarrow c} f(x)$  exists.
3.  $\lim_{x \rightarrow c} f(x) = f(c)$

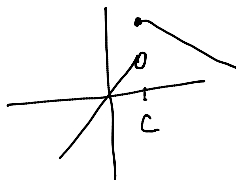
A function is continuous on an open interval  $(a, b)$  if it is continuous at each point in the interval. A function that is continuous on the entire real line  $(-\infty, \infty)$  is everywhere continuous.

Examples: Ways a function can fail to be continuous.

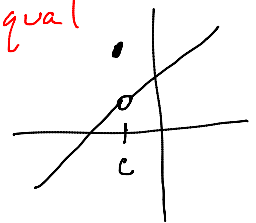
① Not defined



② limit does not exist



③ limit exists,  $f(c)$  is defined, but they are not equal



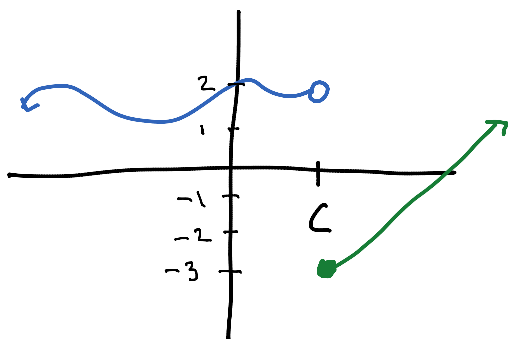
Frequently a function will have different behavior to the left and the right of the value  $c$  under question. In these cases we can look at one-sided limits, limits only from the left or only from the right. Thinking of the number line with negative values to the left and positive values to the right we have the following notation:

1. From the right  $\lim_{x \rightarrow c^+} f(x) = L$

2. From the left  $\lim_{x \rightarrow c^-} f(x) = L$

The Existence of a Limit – Let  $f$  be a function and let  $c$  and  $L$  be real numbers. The limit of  $f(x)$  as  $x$  approaches  $c$  is  $L$  if and only if  $\lim_{x \rightarrow c^-} f(x) = L$  and  $\lim_{x \rightarrow c^+} f(x) = L$ .

In examining a graph to find the limit, we are careful to look at the graph from the left to see what the value of the function is approaching and then from the right of the established value  $c$ .



As  $x \rightarrow c^-$  ( $x$  approaches  $c$  from left) we see that the graph gets close to the  $y$ -value 2.

As  $x \rightarrow c^+$  ( $x$  approaches  $c$  from right) we see that the graph gets close to the  $y$ -value -3.

Since  $\lim_{x \rightarrow c^-} f(x) \neq \lim_{x \rightarrow c^+} f(x)$  the overall limit does not exist (DNE).

Definition of Continuity on a Closed Interval – A function  $f$  is continuous on the closed interval  $[a, b]$  if it is continuous on the open interval  $(a, b)$  and the limit from the right as  $x$  approaches  $a$  is equal to  $f(a)$  and the limit from the left as  $x$  approaches  $b$  is equal to  $f(b)$ .

Examples: Find the limit, if it exists.

$$1. \lim_{x \rightarrow 2^+} \frac{2-x}{x^2-4} = \lim_{x \rightarrow 2^+} \frac{-(x-2)}{(x-2)(x+2)} = \lim_{x \rightarrow 2^+} \frac{-1}{x+2} = \frac{-1}{4}$$

$$2. \lim_{x \rightarrow 10^-} \frac{|x-10|}{x-10}$$

notice that numerator and denominator are the same except possibly in sign. For  $x \rightarrow 10^-$ , the values to the left of 10 are smaller than 10 so  $x-10$  will be negative. This means we will have  $\lim_{x \rightarrow 10^-} \frac{|x-10|}{x-10} = -1$

$$3. \lim_{x \rightarrow \pi} \cot x = \lim_{x \rightarrow \pi} \frac{\cos x}{\sin x} = \frac{-1}{0} \leftarrow \text{cannot divide by 0}$$

so limit DNE

Recall the graph of cotangent to see that  $\lim_{x \rightarrow \pi^+} \cot x \neq \lim_{x \rightarrow \pi^-} \cot x$

4.  $\lim_{x \rightarrow 3^-} f(x)$  where  $f(x) = \begin{cases} \frac{x+2}{2}, & x \leq 3 \\ \frac{12-2x}{3}, & x > 3 \end{cases}$  ← less than is left of

↑  
left of 3

$$\lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3^-} \frac{x+2}{2} = \frac{3+2}{2} = \frac{5}{2}$$

5.  $\lim_{x \rightarrow 4^+} (5\|x\| - 7) = 5(4) - 7 = 13 \rightarrow$  for values close to 4 but slightly above (to the right), the greatest integer yields 4.

↑  
greatest integer function

recall that  $\|4.2\| = 4$ ,  $\|3.9\| = 3$ ,  
 $\|-2.1\| = -3$  and  $\|5\| = 5$

Properties of Continuity – If  $b$  is a real number and  $f$  and  $g$  are continuous at  $x = c$ , then the following functions are also continuous at  $c$ .

1. Scalar Multiple:  $bf$
2. Sum or difference:  $f \pm g$
3. Product:  $fg$
4. Quotient:  $f/g$ , if  $g(x) \neq 0$

This means that most elementary functions are continuous at every point in their domains. The key points to consider are values NOT in the domain.

Examples: Find the constant  $a$  such that the function is continuous on the entire real line.

1.  $f(x) = \begin{cases} 2x^2, & x \geq 1 \\ ax-3, & x < 1 \end{cases}$  To be continuous, the limit from the left must be equal to the limit from the right and be equal to the function value.

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} (ax-3) = a(1)-3 = a-3$$

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} (2x^2) = 2(1)^2 = 2$$

$$\rightarrow a-3=2 \text{ so } \boxed{a=5}$$

Notice that  $\lim_{x \rightarrow 1^+} f(x) = f(1)$ .

$$2. g(x) = \begin{cases} \frac{9\sin x}{x}, & x < 0 \\ a - 8x, & x \geq 0 \end{cases}$$

$$\lim_{x \rightarrow 0^-} g(x) = \lim_{x \rightarrow 0^-} \frac{9\sin x}{x} = 9$$

$$\lim_{x \rightarrow 0^+} g(x) = \lim_{x \rightarrow 0^+} (a - 8x) = a - 8(0) = a$$

It must be that  $a = 9$

Examples: Find the x-values (if any) at which  $f$  is not continuous. Which of the discontinuities are removable.

→ removable if it can be factored out or otherwise dealt with

1.  $f(x) = \frac{3}{x-2}$  Not continuous at  $x=2$  (only look at values not in domain)  
not removable

2.  $f(x) = x^2 - 2x + 1$   
↑  
polynomial so continuous everywhere  
Domain is all real numbers so

3.  $f(x) = \frac{1}{x^2 + 1}$  ← continuous everywhere

4.  $f(x) = \frac{x}{x^2 - 1}$  ← Not continuous at  $x = \pm 1$ , not removable

5.  $f(x) = \frac{x-6}{x^2-36} = \frac{x-6}{(x+6)(x-6)}$  Not continuous at  $x = \pm 6$   
However, the discontinuity at  $x=6$  is removable

6.  $f(x) = \begin{cases} x, & x \leq 1 \\ x^2, & x > 1 \end{cases}$  For piece-wise defined functions we know each piece is continuous on its domain. This means we only have to check where the graph cuts from one piece to the next, the cut point.  
 $\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} x = 1$  and  $\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} x^2 = 1$

Since  $\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^+} f(x) = 1$ , the limit exists and is equal to the function value:  $f(1) = 1$ . This function is continuous everywhere.



Continuity of a Composite Function – If  $g$  is continuous at  $c$  and  $f$  is continuous at  $g(c)$ , then the composite function given by  $(f \circ g)(x)$  is continuous at  $c$ .

**Intermediate Value Theorem** – If  $f$  is continuous on the closed interval  $[a,b]$ ,  $f(a) \neq f(b)$ , and  $k$  is any number between  $f(a)$  and  $f(b)$ , then there is at least one number  $c$  in  $[a,b]$  such that  $f(c) = k$ .

A basic interpretation is that while driving from home to work I start at 0 mph when I get in my car and go 75 mph on the highway. At some point in time I must have been going 55 mph.

Example: Verify that the IVT applies to the indicated interval and find the value of  $c$  guaranteed by the theorem.

So verify 1) cont on the interval

2)  $f(a) \neq f(b)$

3)  $k$  is between  $f(a)$  &  $f(b)$

1.  $f(x) = x^2 - 6x + 8$ ,  $[0,3]$ ,  $f(c) = 0$

①  $f$  is a polynomial so is continuous

②  $f(a) = f(0) = 8$   
 $f(b) = f(3) = -1$  } not equal

③  $k = 0$  is between  $-1$  and  $8$ .

now that these have been verified, we can find the value  $c$ :

$$c^2 - 6c + 8 = 0$$

$$(c-4)(c-2) = 0$$

so  $c = 4, 2$ .

Our solution is  $c = 2$  as it is the only result in  $[0,3]$ .

2.  $f(x) = \frac{x^2 + x}{x-1}$ ,  $[\frac{5}{2}, 4]$ ,  $f(c) = 6$

①  $f$  is cont everywhere except at  $x=1$  which is not in the interval

②  $f(\frac{5}{2}) = \frac{35}{6}$  (a little less than 6)

$f(4) = \frac{20}{3}$  (a little more than 6)

③ 6 is between  $\frac{35}{6}$  and  $\frac{20}{3}$

$$(x-1) \frac{x^2 + x}{x-1} = 6(x-1)$$

$$x^2 + x = 6x - 6$$

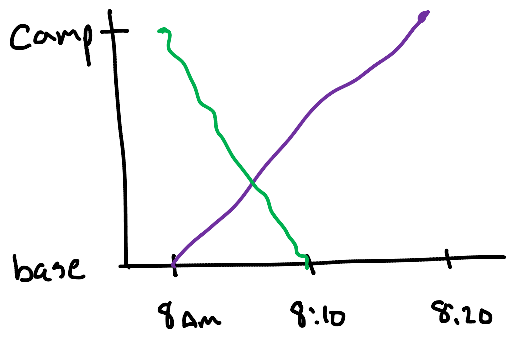
$$x^2 - 5x + 6 = 0$$

$$(x-3)(x-2) = 0$$

$x = 3$  or  $x = 2$

Our answer is  $x = 3$  as the other is not in the given interval

Example: At 8:00 AM on Saturday a man begins running up the side of a mountain to his weekend campsite. On Sunday morning at 8:00 AM he runs back down the mountain. It takes him 20 minutes to run up, but only 10 minutes to run down. At some point on the way down, he realizes that he passed the same place at exactly the same time on Saturday. Prove that he is correct.



Saturday  
Sunday

Notice that in leaving at 8Am both days shows the graphs must intersect.

This intersection is when he was at the same place at the same time both Saturday and Sunday.