Chapter Two: Differentiation

2.1 The Derivative and the Tangent Line Problem

The difference quotient is introduced in pre-calculus as a rate of change. This will be the basis of the definition of derivatives.

Definition of Tangent Line with Slope *m* – If *f* is defined on an open interval containing *c*, and if the limit

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(c+\Delta x) - f(c)}{\Delta x} = m$$

exists, then the line passing through *(c, f(c))* with slope *m* is the tangent line to the graph of *f* at the point *(c, f(c))*.

The slope of the tangent line is also called the slope of the graph.

Examples: Find the slope of the tangent line to the graph of the function at the given point.

1. $f(x) = \frac{3}{2}x + 1, \ (\underline{-2}, -2)$

$c = -2$

$$f(-2+\Delta x) = \frac{3}{2}(-2+\Delta x) + 1$$
$$= -3 + \frac{3}{2}\Delta x + 1$$
$$= -2 + \frac{3}{2}\Delta x$$

$$f(-2) = \frac{3}{2}(-2) + 1 = -3 + 1 = -2$$

$$m = \lim_{\Delta x \to 0} \frac{-2 + \frac{3}{2}\Delta x - (-2)}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{\frac{3}{2}\Delta x}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{3}{2} = \frac{3}{2}$$

The slope of the tangent line to the graph of $f(x) = \frac{3}{2}x + 1$ at $c = -2$ is $\frac{3}{2}$.

Notice that the slope of any linear function is going to also be the slope of the tangent.

2. $g(x)=6-x^2, (1,5)$

$c=1$

$$g(1+\Delta x)=6-(1+\Delta x)^2$$
$$=6-(1+2\Delta x+(\Delta x)^2)$$
$$=6-1-2\Delta x-(\Delta x)^2$$
$$=5-2\Delta x-(\Delta x)^2$$

$$g(1)=6-(1)^2=6-1=5$$

$$m=\lim_{\Delta x\to 0}\frac{5-2\Delta x-(\Delta x)^2-5}{\Delta x}$$
$$=\lim_{\Delta x\to 0}\frac{-2\Delta x-(\Delta x)^2}{\Delta x}$$
$$=\lim_{\Delta x\to 0}\frac{\cancel{\Delta x}(-2-\Delta x)}{\cancel{\Delta x}}$$

factor in order to cancel

$$=\lim_{\Delta x\to 0}(-2-\Delta x)$$
$$=-2-0=-2$$

3. $h(t)=t^2+3, (-2,7)$

$c=-2$

$$h(-2+\Delta t)=(-2+\Delta t)^2+3$$
$$=(4-4\Delta t+(\Delta t)^2)+3$$
$$=7-4\Delta t+(\Delta t)^2$$

$$h(-2)=(-2)^2+3=4+3=7$$

$$m=\lim_{\Delta t\to 0}\frac{7-4\Delta t+(\Delta t)^2-7}{\Delta t}$$
$$=\lim_{\Delta t\to 0}\frac{-4\Delta t+(\Delta t)^2}{\Delta t}$$
$$=\lim_{\Delta t\to 0}\frac{\Delta t(-4+\Delta t)}{\Delta t}$$
$$=\lim_{\Delta t\to 0}(-4+\Delta t)=-4$$

Definition of the Derivative of a Function – The derivative of f at x is given by

$$f'(x)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

provided the limits exists. For all x for which this limit exists, f' is a function of x.

Notation – The following are equivalent: $f'(x), \frac{dy}{dx}, y', \frac{d}{dx}[f(x)], D_x[y]$

Examples: Find the derivative by the limit process.

1. $f(x) = 3x + 2$

$$f(x+\Delta x) = 3(x+\Delta x) + 2 = 3x + 3\Delta x + 2$$

$$\begin{array}{rl} f(x+\Delta x) & 3x + 3\Delta x + 2 \\ - f(x) & -(3x \qquad +2) \\ \hline f(x+\Delta x) - f(x) = & 3\Delta x \end{array}$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{3\Delta x}{\Delta x} = \lim_{\Delta x \to 0} 3 = 3$$

That is, $f'(x) = 3$.

2. $g(x) = 2 - x^2$

$$g(x+\Delta x) = 2 - (x+\Delta x)^2 = 2 - (x^2 + 2x\Delta x + (\Delta x)^2) = 2 - x^2 - 2x\Delta x - (\Delta x)^2$$

$$g(x+\Delta x) - g(x) = -2x\Delta x - (\Delta x)^2$$

$$g'(x) = \lim_{\Delta x \to 0} \frac{-2x\Delta x - (\Delta x)^2}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x(-2x - \Delta x)}{\Delta x} = \lim_{\Delta x \to 0} (-2x - \Delta x) = -2x$$

3. $f(x) = \dfrac{4}{\sqrt{x}}$

$$f(x+\Delta x) = \frac{4}{\sqrt{x+\Delta x}}$$

too easy, right?

$$f(x+\Delta x) - f(x) = \frac{4}{\sqrt{x+\Delta x}} - \frac{4}{\sqrt{x}}$$

this needs work

$$= \frac{4}{\sqrt{x+\Delta x}} \cdot \frac{\sqrt{x}}{\sqrt{x}} - \frac{4}{\sqrt{x}} \cdot \frac{\sqrt{x+\Delta x}}{\sqrt{x+\Delta x}}$$

$$= \frac{4\sqrt{x}}{\sqrt{x}\sqrt{x+\Delta x}} - \frac{4\sqrt{x+\Delta x}}{\sqrt{x}\sqrt{x+\Delta x}}$$

this still isn't enough because...

$$f'(x) = \lim_{\Delta x \to 0} \frac{\frac{4\sqrt{x} - 4\sqrt{x+\Delta x}}{\sqrt{x}\sqrt{x+\Delta x}}}{\Delta x}$$

... is still undefined *

* more algebra *

$$\frac{4\sqrt{x} - 4\sqrt{x+\Delta x}}{\sqrt{x}\sqrt{x+\Delta x}} \cdot \frac{4\sqrt{x} + 4\sqrt{x+\Delta x}}{4\sqrt{x} + 4\sqrt{x+\Delta x}}$$

$$= \frac{16x - 16(x+\Delta x)}{\sqrt{x}\sqrt{x+\Delta x}(4\sqrt{x} + 4\sqrt{x+\Delta x})}$$

$$= \frac{-16\Delta x}{\sqrt{x}\sqrt{x+\Delta x}(4\sqrt{x} + 4\sqrt{x+\Delta x})}$$

$$= \lim_{\Delta x \to 0} \frac{\frac{-16\Delta x}{\sqrt{x}\sqrt{x+\Delta x}(4\sqrt{x} + 4\sqrt{x+\Delta x})}}{\Delta x} = \lim_{\Delta x \to 0} \frac{-16}{\sqrt{x}\sqrt{x+\Delta x}(4\sqrt{x} + 4\sqrt{x+\Delta x})}$$

$$= \frac{-16}{\sqrt{x}\sqrt{x}(4\sqrt{x} + 4\sqrt{x})} = \frac{-16}{8x\sqrt{x}}$$

And finally, $f'(x) = \dfrac{-2}{x\sqrt{x}}$

Examples: Find an equation of the tangent line to the graph of f at the given point. $\to y - f(c) = f'(c)(x-c)$

1. $f(x) = x^2 + 3x + 4$, $(-2, 2)$

c $f(c)$

$$f(x+\Delta x) = (x+\Delta x)^2 + 3(x+\Delta x) + 4$$
$$= x^2 + 2x\Delta x + (\Delta x)^2 + 3x + 3\Delta x + 4$$
$$f(x+\Delta x) - f(x) = 2x\Delta x + (\Delta x)^2 + 3\Delta x$$
$$\frac{f(x+\Delta x) - f(x)}{\Delta x} = 2x + \Delta x + 3$$

$$f'(x) = \lim_{\Delta x \to 0} (2x + \Delta x + 3) = 2x + 3$$

so $f'(-2) = 2(-2) + 3 = -4 + 3 = -1$

Tangent line

$$y - 2 = -1(x - (-2))$$
$$y - 2 = -x - 1$$
$$y = -x + 1$$

2. $f(x) = \sqrt{x-1}$, $(5, 2)$

$$f(5+\Delta x) = \sqrt{5 + \Delta x - 1} = \sqrt{4 + \Delta x}$$
$$f(5) = \sqrt{5-1} = \sqrt{4} = 2$$
$$f(5+\Delta x) - f(5) = \sqrt{4+\Delta x} - 2$$
$$\frac{f(5+\Delta x) - f(5)}{\Delta x} = \frac{\sqrt{4+\Delta x} - 2}{\Delta x} \cdot \frac{\sqrt{4+\Delta x} + 2}{\sqrt{4+\Delta x} + 2}$$
$$= \frac{4 + \Delta x - 4}{(\sqrt{4+\Delta x} + 2)\Delta x} = \frac{1}{\sqrt{4+\Delta x} + 2}$$

$$f'(5) = \lim_{\Delta x \to 0} \frac{1}{\sqrt{4+\Delta x} + 2}$$
$$= \frac{1}{\sqrt{4} + 2}$$
$$= \frac{1}{4} = m$$

Tangent line is $y - 2 = \frac{1}{4}(x - 5)$

$$y - 2 = \frac{1}{4}x - \frac{5}{4}$$
$$y = \frac{1}{4}x + \frac{3}{4}$$

3. $f(x) = \dfrac{1}{x+1}$, $(0, 1)$

$$f(0+\Delta x) = \frac{1}{\Delta x + 1}$$
$$f(0) = 1$$
$$f(0+\Delta x) - f(0) = \frac{1}{\Delta x + 1} - 1$$
$$= \frac{1}{\Delta x + 1} - \frac{\Delta x + 1}{\Delta x + 1}$$
$$= \frac{-\Delta x}{\Delta x + 1}$$

$$f'(0) = \lim_{\Delta x \to 0} \frac{\frac{-\Delta x}{\Delta x + 1}}{\Delta x} = \lim_{\Delta x \to 0} \frac{-\Delta x}{\Delta x + 1} \cdot \frac{1}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{-1}{\Delta x + 1} = -1$$

$$y - 1 = -1(x - 0)$$
$$y - 1 = -x$$
$$y = -x + 1$$

Example: Find an equation of the line that is tangent to $f(x)=x^3+2$ and is parallel to $3x-y-4=0$.

same slope

$3x-4=y$

so $m=3$

$f(x+\Delta x)=(x+\Delta x)^3+2$

$= x^3+3x^2\Delta x+3x(\Delta x)^2+(\Delta x)^3+2$

$f(x+\Delta x)-f(x)=3x^2\Delta x+3x(\Delta x)^2+(\Delta x)^3$

We must find a point on $f(x)$ where $f'(x)=3$

First we find $f'(x)$.

$$\frac{f(x+\Delta x)-f(x)}{\Delta x}=3x^2+3x\Delta x+(\Delta x)^2$$

$$f'(x)=\lim_{\Delta x\to 0}\left(3x^2+3x\Delta x+(\Delta x)^2\right)=3x^2$$

If $f'(x)=3x^2$, then $f'(x)=3$ when

$3x^2=3$ or $x=\pm 1$.

This gives two tangent lines, one through $(-1,1)$ and one at $(1,3)$

$y-1=3(x-(-1))$

$y-1=3x+3$

$\boxed{y=3x+4}$

$y-3=3(x-1)$

$y-3=3x-3$

$\boxed{y=3x}$

Verify both are accurate using a graphing utility.

Alternative Definition of the Derivative – The derivative of *f* at *c* is

$$f'(\cancel{x}\,c)=\lim_{x\to c}\frac{f(x)-f(c)}{x-c}$$

provided this limit exists. Notice that this quotient is just the formula for the slope of a line between two points and the limit is what makes it work for nonlinear functions.

Examples: Use the alternative form of the derivative to find the derivative at $x = c$, if it exists.

1. $f(x)=x(x-1),\ c=1$

$$f'(\cancel{x}\,1)=\lim_{x\to 1}\frac{x(x-1)-1(1-1)}{x-1}=\lim_{x\to 1}\frac{x\cancel{(x-1)}}{\cancel{x-1}}=\lim_{x\to 1}x=1$$

$f'(1)$ not $f'(x)$ in general

So $f'(1)=1$

2. $f(x)=\frac{2}{x}$, $c=5$

$$f'(5)=\lim_{x\to 5}\frac{\frac{2}{x}-\frac{2}{5}}{x-5}=\lim_{x\to 5}\frac{-2}{5x}=\frac{-2}{5(5)}=\frac{-2}{25}$$

a) $$\frac{2}{x}-\frac{2}{5}=\frac{2}{x}\cdot\frac{5}{5}-\frac{2}{5}\cdot\frac{x}{x}=\frac{10}{5x}-\frac{2x}{5x}=\frac{10-2x}{5x}$$

b) $$\frac{\frac{10-2x}{5x}}{x-5}=\frac{10-2x}{5x}\cdot\frac{1}{x-5}=\frac{2(\overset{-1}{\cancel{5-x}})}{5x}\cdot\frac{1}{\cancel{x-5}}=\frac{-2}{5x}$$