2.3 Product and Quotient Rules and Higher-Order Derivatives

The Product Rule – The product of two differentiable functions *f* and *g* is itself differentiable. Moreover, the derivative of *fg* is the first function multiplied by the derivative of the second, plus the second function multiplied by the derivative of the first.

$$\frac{d}{dx}\left[f(x)g(x)\right] = f(x)g'(x) + g(x)f'(x)$$

Examples: Use the product rule to find the derivative.

1. $f(x) = \underbrace{(6x+5)}_{1^{st}}\underbrace{(x^3-2)}_{2^{nd}}$

$f'(x) = \underbrace{(6x+5)}_{1^{st}}\underbrace{(3x^2-0)}_{d\,2^{nd}} \underset{+}{+} \underbrace{(x^3-2)}_{2^{nd}}\underbrace{(6+0)}_{d\,1^{st}} = (6x+5)(3x^2) + (x^3-2)(6)$

$f'(x) = 18x^3 + 15x^2 + 6x^3 - 12 = 24x^3 + 15x^2 - 12$

2. $g(x) = \underbrace{\sqrt{x}}_{1^{st}}\underbrace{\sin x}_{2^{nd}} = x^{1/2}\sin x$

$g'(x) = \underbrace{\sqrt{x}}_{1^{st}}\underbrace{\cos x}_{d\,2^{nd}} + \underbrace{\sin x}_{2^{nd}}\underbrace{\left(\frac{1}{2}x^{-\frac{1}{2}}\right)}_{d\,1^{st}} = \sqrt{x}\cos x + \frac{\sin x}{2\sqrt{x}}$

3. $h(x) = \underbrace{\left(x^{-2} + \frac{1}{x}\right)}_{1^{st}}\underbrace{\left(\sqrt[3]{x} - \cos x\right)}_{2^{nd}} = (x^{-2} + x^{-1})(x^{1/3} - \cos x)$

$h'(x) = \left(\frac{1}{x^2} + \frac{1}{x}\right)\left(\frac{1}{3}x^{-2/3} + \sin x\right) + \left(\sqrt[3]{x} - \cos x\right)\left(-2x^{-3} - 1x^{-2}\right)$

$= \left(\frac{1}{x^2} + \frac{1}{x}\right)\left(\frac{1}{3x^{2/3}} + \sin x\right) + \left(\sqrt[3]{x} - \cos x\right)\left(\frac{-2}{x^3} - \frac{1}{x^2}\right)$

Sometimes simplifying is just ridiculous! Let's leave this one.

$1^{st}$ $2^{nd}$ $3^{rd}$ ?!?

4. $y=\left(x^2+3x\right)\left(2x-1\right)\left(x^5-\sin x\right) = \underbrace{(x^2+3x)}_{1^{st}}\underbrace{\left[(2x-1)(x^5-\sin x)\right]}_{2^{nd}}$ der 2nd will use product rule again

$$y' = (x^2+3x)\frac{d}{dx}\left[(2x-1)(x^5-\sin x)\right] + (2x-1)(x^5-\sin x)\frac{d}{dx}(x^2+3x)$$

$$= \underbrace{(x^2+3x)}_{1^{st}}\underbrace{\left[(2x-1)(5x^4-\cos x) + (x^5-\sin x)(2)\right]}_{d\ 2^{nd}} \underbrace{+}_{+} \underbrace{(2x-1)(x^5-\sin x)}_{2^{nd}}\underbrace{(2x+3)}_{d\ 1^{st}}$$

we'll leave this one as is too

The Quotient Rule – The quotient *f/g* of two differentiable functions *f* and *g* is itself differentiable at all values of *x* for with $g(x)\neq 0$. Moreover, the derivative of *f/g* is given by the denominator multiplied by the derivative of the numerator minus the numerator multiplied by the derivative of the denominator, all divided by the square of the denominator. Alternatively, lo de hi minus hi de lo over lo squared.

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]=\frac{g(x)f'(x)-f(x)g'(x)}{\left[g(x)\right]^2},\ g(x)\neq 0$$

$$\frac{d}{dx}\left[\frac{hi}{lo}\right] = \frac{lo\,dehi - hi\,delo}{lo^2}$$

Examples: Use the quotient rule to find the derivative.

1. $g(t)=\dfrac{t^2+4}{5t-3}$ $\quad g'(t) = \dfrac{(5t-3)(2t+0)-(t^2+4)(5-0)}{(5t-3)^2} = \dfrac{10t^2-6t-5t^2-20}{(5t-3)^2}$

or $g'(t) = \dfrac{5t^2-6t-20}{(5t-3)^2}$

parenthesis are very important

as is the order of subtraction in the numerator

2. $h(x)=\dfrac{x}{\sqrt{x}-1}$

$$h'(x)=\frac{(\sqrt{x}-1)(1)-x\left(\frac{1}{2}x^{-\frac{1}{2}}-0\right)}{(\sqrt{x}-1)^2}=\frac{\sqrt{x}-1-\frac{x}{2\sqrt{x}}}{(\sqrt{x}-1)^2}$$

3. $f(x)=\dfrac{\sin x}{x^3}$

$$f'(x)=\frac{x^3\cos x-\sin x(3x^2)}{(x^3)^2}=\frac{x^3\cos x-3x^2\sin x}{x^6}$$

4. $y=\dfrac{x^3-2x^2+6x^{-4}}{5x^8+\sin x}$

$$y'=\frac{(5x^8+\sin x)(3x^2-4x-24x^{-5})-(x^3-2x^2+6x^{-4})(40x^7+\cos x)}{(5x^8+\sin x)^2}$$

yep... that looks pretty simplified to me :-)

Examples: Find the derivatives using the product and quotient rules.

1. $f(x)=(x^7+3x^4)\left(\dfrac{\sin x}{x}\right)$

$$f'(x)=(x^7+3x^4)\cdot\frac{d}{dx}\left(\frac{\sin x}{x}\right)+\frac{\sin x}{x}\cdot\frac{d}{dx}(x^7+3x^4)$$

$$f'(x)=(x^7+3x^4)\left[\frac{x\cos x-\sin x(1)}{x^2}\right]+\frac{\sin x}{x}(7x^6-12x^3)$$

This can be simplified quite easily, but we will leave it here

2. $y = \dfrac{(\sin x)(3x^{-2}+5x)}{\cos x - 7}$

$$y' = \frac{(\cos x - 7)\frac{d}{dx}\left[(\sin x)(3x^{-2}+5x)\right] - (\sin x)(3x^{-2}+5x)\frac{d}{dx}(\cos x - 7)}{(\cos x - 7)^2}$$

$$y' = \frac{(\cos x - 7)\left[(\sin x)(-6x^{-3}+5) + (3x^{-2}+5x)(\cos x)\right] - (\sin x)(3x^{-2}+5x)(-\sin x)}{(\cos x - 7)^2}$$

Derivatives of Trigonometric Functions – Using the quotient rule we can now find the derivatives of the remaining trig functions

$$\frac{d}{dx}[\tan x] = \sec^2 x \qquad \frac{d}{dx}\left[\cot x = -\csc^2 x\right]$$

$$\frac{d}{dx}[\sec x] = \sec x \tan x \qquad \frac{d}{dx}[\csc x] = -\csc x \cot x$$

I'll show $\frac{d}{dx}(\tan x) = \sec^2 x$, you should verify the others

Let $y = \tan x = \dfrac{\sin x}{\cos x}$. Then $y' = \dfrac{\cos x(\cos x) - \sin x(-\sin x)}{(\cos x)^2}$. Now

simplify $y' = \dfrac{\cos^2 x + \sin^2 x}{\cos^2 x} = \dfrac{1}{\cos^2 x} = \sec^2 x$.

Higher Order Derivatives – Just as we can find the derivative of a position function to find a velocity function, we can find the derivative of the velocity function (since it is just a function) to find the acceleration function. Though direct applications may run out after the third derivative (the jerk function), we can take derivatives as long as we may want to. We continue to use tick marks up to the third derivative and then switch the notation to a subscript number in parentheses: $\dfrac{d^4y}{dx^4} = f^{(4)}(x)$.

Examples: Find the derivatives of the trig functions.

1. $f(\theta)=(\theta+1)\cos\theta$

$$f'(\theta)=(\theta+1)(-\sin\theta)+\cos\theta(1)$$
$$=-\theta\sin\theta-\sin\theta+\cos\theta$$

2. $y=x+\cot x$

$$y'=1-\csc^2 x$$

3. $y=\dfrac{\sec x}{x}$

$$y'=\frac{x(\sec x\tan x)-\sec x(1)}{x^2}=\frac{x\sec x\tan x-\sec x}{x^2}=\frac{\sec x(x\tan x-1)}{x^2}$$

all three representations are acceptable

4. $y=\overset{\text{P.R.}}{x\sin x}+\csc x$

$$y'=x\cos x+\sin x(1)-\csc x\cot x$$

5. $h(\theta)=5\theta\sec\theta+\theta\tan\theta$

$$h'(\theta)=5\theta(\sec\theta\tan\theta)+\sec\theta(5)+\theta\sec^2\theta+\tan\theta(1)$$
$$h'(\theta)=5\theta\sec\theta\tan\theta+5\sec\theta+\theta\sec^2\theta+\tan\theta$$

Example: Find equations of the tangent lines to the graph of $f(x)=\dfrac{x+1}{x-1}$ that are parallel to the line $2y+x=6$.

$2y=-x+6$

$y=-\frac{1}{2}x+3$

$m=-\frac{1}{2}$

Find $f'(x)=-\frac{1}{2}$

$$f'(x)=\frac{(x-1)(1)-(x+1)(1)}{(x-1)^2}=\frac{x-1-x-1}{(x-1)^2}=\frac{-2}{(x-1)^2}$$

$\dfrac{-2}{(x-1)^2}=-\dfrac{1}{2}$ means $(x-1)^2=4$

$x-1=\pm\sqrt{4}=\pm 2$

$x-1=2$, $x-1=-2$

$x=3$ $\quad$ $x=-1$

Now we know our x-values, we can find the corresponding y-values: $x=3$: $y=\frac{3+1}{3-1}=\frac{4}{2}=2$

$x=-1$: $y=\frac{-1+1}{-1-1}=\frac{0}{-2}=0$

Finally we can find the tangent lines

$(3,2)$: $y-2=\frac{-1}{2}(x-3)$

$y-2=\frac{1}{2}x+\frac{3}{2}$

$y=-\frac{1}{2}x+\frac{7}{2}$

$(-1,0)$: $y-0=-\frac{1}{2}(x-(-1))$

$y-0=-\frac{1}{2}(x+1)$

$y=-\frac{1}{2}x-\frac{1}{2}$

Example: An automobile's velocity starting from rest is $v(t)=\dfrac{100t}{2t+15}$ where $v$ is measured in feet per second. Find the acceleration at (a) 5 seconds, (b) 10 seconds, and (c) 20 seconds.

Acceleration is the derivative of velocity (and second der. of position).

$$a(t)=v'(t)=\frac{(2t+15)(100)-100t(2)}{(2t+15)^2}=\frac{200t+1500-200t}{(2t+15)^2}$$

$$a(t)=\frac{1500}{(2t+15)^2}$$

a) at $t=5$ seconds, $a(5)=\dfrac{1500}{(2(5)+15)^2}=\dfrac{1500}{25^2}=\dfrac{12}{5}=2.4$ ft/sec²

b) at $t=10$ seconds, $a(10)=\dfrac{1500}{(2(10)+5)^2}=\dfrac{1500}{35^2}=\dfrac{60}{49}\approx 1.22$ ft/sec²

c) at $t=20$ seconds $a(20)=\dfrac{1500}{(2(20)+15)^2}=\dfrac{1500}{55^2}=\dfrac{60}{121}\approx 0.50$ ft/sec²