2.4 The Chain Rule

In order to completely understand the chain rule, we must first revisit composition of functions. Composition of functions is evaluating one function with another and is frequently written $y = f\left(g\left(x\right)\right)$. I will refer to *f* as the “outside” function and *g* as the “inside” function.

Examples: For the given function, determine the inside function $u = g\left(x\right)$ and the outside function $y = f\left(u\right)$.

1. $y = \frac{1}{\sqrt{x+1}}$

   It looks like $x+1$ is inside a radical so maybe $u = x+1$ and $y = \frac{1}{\sqrt{u}}$.

   There are many other possibilities, always go simple.

2. $y = 3\tan\left(\pi x^2\right)$

   $u = \pi x^2$

   $y = 3\tan(u)$

3. $y = \sqrt{x^3 - 7}$

   $u = x^3 - 7$

   $y = \sqrt{u}$

4. $y = \sin\frac{5x}{2}$

   $u = \frac{5x}{2}$

   $y = \sin u$

The Chain Rule – If $y = f\left(u\right)$ is a differentiable function of *u* and $u = g\left(x\right)$ is a differentiable function of *x*, then $y = f\left(g\left(x\right)\right)$ is a differentiable function of *x* and $\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$ or, equivalently, $\frac{d}{dx}\left[f\left(g\left(x\right)\right)\right] = f'\left(g\left(x\right)\right)g'\left(x\right)$.

The General Power Rule – If $y=\left[u(x)\right]^n$, where u is a differentiable function of x and n is a rational number, then $\frac{dy}{dx}=n\left[u(x)\right]^{n-1}\frac{du}{dx}$ or, equivalently, $\frac{d}{dx}\left[u^n\right]=nu^{n-1}u'$.

Examples: Find the derivative of the function.

1. $y=2\left(6-x^2\right)^5$

inside $u=6-x^2$

outside $y=2u^5$

$y'=2(5u^4)\cdot(0-2x)$

$y'=10(6-x^2)^4(-2x)$

$y'=-20x(6-x^2)^4$

2. $g(x)=\sqrt{x^2-2x+1}$

Inside $u=x^2-2x+1$

outside $y=\sqrt{u}$

$y'=\frac{1}{2}u^{-1/2}\cdot(2x-2+0)$

$y'=\frac{1}{2}(x^2-2x+1)^{-1/2}(2x-2)$

$y'=\frac{2x-2}{2\sqrt{x^2-2x+1}}=\frac{x-1}{\sqrt{x^2-2x+1}}=g'(x)$

3. $y=-\frac{5}{(t+3)^3}$

In $u=t+3$

out $y=-\frac{5}{u^3}=-5u^{-3}$

$y'=-5(-3u^{-4})\cdot(1+0)$

$y'=15(t+3)^{-4}(1)$

$y'=\frac{15}{(t+3)^4}$

Quotient with chain rule in denom.

4. $y = \dfrac{x}{\sqrt{x^4+4}}$

$$y' = \frac{\sqrt{x^4+4}\,(1) - x\frac{d}{dx}\sqrt{x^4+4}}{\left(\sqrt{x^4+4}\right)^2}$$

$$= \frac{\sqrt{x^4+4} - x\left(\frac{2x^3}{\sqrt{x^4+4}}\right)}{x^4+4}$$

not exactly simplified, but we shall leave it here anyway.

$u = x^4+4$

$y = \sqrt{u}$

$y' = \frac{1}{2}u^{-1/2}(4x^3+0)$

$= \dfrac{4x^3}{2\sqrt{x^4+4}} = \dfrac{2x^3}{\sqrt{x^4+4}}$

5. $h(t) = \left(\dfrac{t^2}{t^3+2}\right)^2$

$$h'(t) = 2u\left(\frac{(t^3+2)(2t) - t^2(3t^2)}{(t^3+2)^2}\right)$$

$$h'(t) = \frac{2t^2}{t^3+2}\left(\frac{2t^4+4t-3t^4}{(t^3+2)^2}\right)$$

$$h'(t) = \frac{2t^2(4t-t^4)}{(t^3+2)^3}$$

$u = \dfrac{t^2}{t^3+2}$

$y = u^2$

6. $f(x) = \left(2+\left(x^2+1\right)^4\right)^3$

This is a repeated chain rule. Let's take it step by step.

$$f'(x) = 3\left(2+(x^2+1)^4\right)^2 \cdot \frac{d}{dx}\left(2+(x^2+1)^4\right)$$

der of outside $3(\text{in})^2$, leave inside alone, times deriv. of in

$$= 3\left(2+(x^2+1)^4\right)^2 \cdot \left(0 + 4(x^2+1)^3 \cdot \frac{d}{dx}(x^2+1)\right)$$

Chain rule again.

$$= 3\left(2+(x^2+1)^4\right)^2\left(4(x^2+1)^3(2x)\right)$$

let's just walk away from simplifying...

7. $y = \sin \pi x$

$u = \pi x$

$y = \sin u$

$y' = \cos u(\pi)$

$y' = \pi \cos(\pi x)$

8. $y = \cos(1-2x)^2$

$v = 1-2x$

$u = v^2$

$y = \cos u$

$y' = -\sin u \frac{du}{dv}$

$y' = -\sin v^2 (2v)\frac{dv}{dx}$

$y' = -\sin(1-2x)^2(2(1-2x))(-2)$

$y' = 4(1-2x)\sin(1-2x)^2$

9. $y = 3x - 5\cos(\pi x)^2$

$y' = 3 - 5\left(-\sin(\pi x)^2 \cdot \frac{d}{dx}(\pi x)^2\right)$

$= 3 + 5\sin(\pi x)^2\left(2(\pi x)\frac{d}{dx}(\pi x)\right)$

$= 3 + 5\sin(\pi x)^2(2\pi x \cdot \pi)$

$y' = 3 + 10\pi^2 x \sin(\pi x^2)$

Examples: Find an equation of the tangent line to the graph of *f* at the given point.

$y - f(c) = f'(c)(x-c)$

1. $y = (4x^3+3)^2 \quad (-1,1)$

$y' = 2(4x^3+3)(12x^2)$

$y'(-1) = 2(4(-1)^3+3)(12(-1)^2)$

$= 2(-1)(12)$

$= -24$

$y - 1 = -24(x-(-1))$

$y - 1 = -24(x+1)$

$y - 1 = -24x - 24$

$\boxed{y = -24x - 23}$

2. $f(x)=\tan^2 x \quad \left(\frac{\pi}{4},1\right)$

$$f(x)=(\tan x)^2$$

$$f'(x)=2(\tan x)(\sec^2 x)$$

$$f'\left(\tfrac{\pi}{4}\right)=2\left(\tan\tfrac{\pi}{4}\right)\left(\sec\tfrac{\pi}{4}\right)^2$$

$$=2(1)(2)=4$$

$$y-1=4\left(x-\tfrac{\pi}{4}\right)$$

$$y-1=4x-\pi$$

$$y=4x-\pi+1$$

Example: The frequency F of a fire truck siren heard by a stationary observer is $F=132,400/(331\pm v)$, where $\pm v$ represents the velocity of the accelerating fire truck in meters per second. Find the rate of change of F with respect to v when rate of change => derivative

a) the fire truck is approaching at a velocity of 30 meters per second (use $-v$).

$$F=\frac{132,400}{331-v} \qquad \frac{dF}{dv}=\frac{(331-v)(0)-132,400(0-1)}{(331-v)^2}=\frac{132,400}{(331-v)^2}$$

$$\left.\frac{dF}{dv}\right|_{v=30}=\frac{132,400}{(331-30)^2}=\frac{132,400}{(301)^2}\approx 1.46 \text{ Hz/m/s}$$

b) the fire truck is moving away at a velocity of 30 meters per second (use $+v$).

$$F=\frac{132,400}{331+v} \qquad \frac{dF}{dv}=\frac{(331+v)(0)-132,400(0+1)}{(331+v)^2}=\frac{-132,400}{(331+v)^2}$$

$$\left.\frac{dF}{dv}\right|_{v=30}=\frac{-132,400}{(331+30)^2}=\frac{-132,400}{(361)^2}\approx -1.02 \text{ Hz/m/s}$$