2.5 Implicit Differentiation

An explicit function is a function that explicitly tells you how to find one of the variable values such as $y = f(x)$. An implicit function is less direct in that no variable has been isolated and in many cases it cannot be isolated. An example might be $xy = 6$ or $x^2 - xy + y^2 - 4 = 0$. In the first example, we could isolate either variable easily. In the second example it is not easy to isolate either variable (possible but not easy).

Guidelines for Implicit Differentiation –

1. Differentiate both sides of the equation *with respect to x.*

2. Collect all terms involving *dy/dx* on the left side of the equation and move all other terms to the right side of the equation.

3. Factor *dy/dx* out of the left side of the equation.

4. Solve for *dy/dx*.

Examples: Find *dy/dx* by implicit differentiation.

1. $x^2 - y^2 = 25$

Always remember $y = y(x)$. So anytime you take the derivative of y you get $y' = \frac{dy}{dx}$.

Derivative of x^2 is $2x$
Derivative of y^2 is $2y\frac{dy}{dx}$ (chain rule)
Derivative of 25 is 0

Derivative of $x^2 - y^2 = 25$ is $2x - 2y\frac{dy}{dx} = 0$. Now solve for $\frac{dy}{dx}$:

$$2x = 2y\frac{dy}{dx} \rightarrow \frac{2x}{2y} = \frac{dy}{dx} \rightarrow \boxed{\frac{dy}{dx} = \frac{x}{y}}$$

2. $x^3 + y^3 = 64$

$$3x^2 + 3y^2\frac{dy}{dx} = 0$$

$$3y^2\frac{dy}{dx} = -3x^2$$

$$\frac{dy}{dx} = \frac{-3x^2}{3y^2} = \frac{-x^2}{y^2}$$

3. $x^2y + y^2x = -2$

Product

Product

$$x^2\left(\frac{dy}{dx}\right) + y(2x) + y^2(1) + x\left(2y\frac{dy}{dx}\right) = 0$$

$$x^2\frac{dy}{dx} + 2xy + y^2 + 2xy\frac{dy}{dx} = 0$$

$$x^2\frac{dy}{dx} + 2xy\frac{dy}{dx} = -y^2 - 2xy$$

$$\frac{dy}{dx}(x^2 + 2xy) = -y^2 - 2xy$$

$$\frac{dy}{dx} = \frac{-y^2 - 2xy}{x^2 + 2xy}$$

4. $\cot y = x - y$

$$-\csc^2 y\frac{dy}{dx} = 1 - \frac{dy}{dx}$$

$$\frac{dy}{dx} - \csc^2 y\frac{dy}{dx} = 1$$

$$\frac{dy}{dx}(1 - \csc^2 y) = 1$$

$$\boxed{\frac{dy}{dx} = \frac{1}{1 - \csc^2 y}}$$

Examples: Find two explicit functions by solving the equation for *y* in terms of *x*.

1. $x^2 + y^2 = 64$

$y^2 = 64 - x^2$

$\sqrt{y^2} = \pm\sqrt{64 - x^2}$

$y = +\sqrt{64 - x^2}$ and $y = -\sqrt{64 - x^2}$

To find $\frac{dy}{dx}$ at this point, we would have to find two derivatives. Implicit make it easier

2. $16y^2 - x^2 = 16$

$16y^2 = 16 + x^2$

$y^2 = \frac{16 + x^2}{16}$

$y = \pm\sqrt{\frac{16 + x^2}{16}}$

$y = \frac{\sqrt{16 + x^2}}{4}$ and $y = \frac{-\sqrt{16 + x^2}}{4}$

Examples: Find *dy/dx* by implicit differentiation and evaluate the derivative at the given point.

1. $xy = 6, \ (-6, -1)$ Product rule

$x\frac{dy}{dx} + y(1) = 0$

$x\frac{dy}{dx} + y = 0$

$x\frac{dy}{dx} = -y$

$\frac{dy}{dx} = \frac{-y}{x}$

$\left.\frac{dy}{dx}\right|_{(-6,-1)} = \frac{-(-1)}{-6} = -\frac{1}{6}$

2. $(x + y)^3 = x^3 + y^3 \quad (-1, 1)$

$3(x+y)^2\left(1 + \frac{dy}{dx}\right) = 3x^2 + 3y^2\frac{dy}{dx}$

$3(x+y)^2 + 3(x+y)^2\frac{dy}{dx} = 3x^2 + 3y^2\frac{dy}{dx}$

$3(x+y)^2\frac{dy}{dx} - 3y^2\frac{dy}{dx} = 3x^2 - 3(x+y)^2$

$\frac{dy}{dx}\left[3(x+y)^2 - 3y^2\right] = 3x^2 - 3(x+y)^2$

$\frac{dy}{dx} = \frac{3x^2 - 3(x+y)^2}{3(x+y)^2 - 3y^2}$

at $(-1,1)$

$\frac{dy}{dx} = \frac{3(-1)^2 - 3(-1+1)^2}{3(-1+1)^2 - 3(1)^2}$

$\frac{dy}{dx} = \frac{3 - 0}{0 - 3}$

$\frac{dy}{dx} = \frac{3}{-3} = -1$

Examples: Find d^2y/dx^2 in terms of *x* and *y*. — first find $\frac{dy}{dx}$ then take the derivative again.

1. $x^2y^2 - 2x = 3$

$$x^2\left(2y\frac{dy}{dx}\right) + y^2(2x) - 2 = 0$$

$$2x^2y\frac{dy}{dx} + 2xy^2 - 2 = 0$$

$$2x^2y\frac{dy}{dx} = 2 - 2xy^2$$

$$\frac{dy}{dx} = \frac{2-2xy^2}{2x^2y} = \frac{1-xy^2}{x^2y}$$

To find $\frac{d^2y}{dx^2}$ we use the quotient rule with some product rule thrown in.

$$\frac{d^2y}{dx^2} = \frac{x^2y\left(0-\left(x2y\frac{dy}{dx} + y^2(1)\right)\right) - (1-xy^2)\left[x^2\frac{dy}{dx} + y(2x)\right]}{(x^2y)^2}$$

We clean it up by substituting $\frac{dy}{dx}$ as indicated.

$$\frac{d^2y}{dx^2} = \frac{-x^2y\left(2xy\frac{1-xy^2}{x^2y} + y^2\right) - (1-xy^2)\left(x^2\frac{1-xy^2}{x^2y} + 2xy\right)}{x^4y^2}$$

2. $1 - xy = x - y$

$$0 - \left(x\frac{dy}{dx} + y(1)\right) = 1 - \frac{dy}{dx}$$

$$-x\frac{dy}{dx} - y = 1 - \frac{dy}{dx}$$

$$\frac{dy}{dx} - x\frac{dy}{dx} = 1 + y$$

$$\frac{dy}{dx}(1-x) = 1+y \quad \text{so} \quad \frac{dy}{dx} = \frac{1+y}{1-x}$$

$$\frac{d^2y}{dx^2} = \frac{(1-x)\frac{dy}{dx} - (1+y)(-1)}{(1-x)^2} = \frac{(1-x)\frac{1+y}{1-x} + (1+y)}{(1-x)^2} = \frac{1+y+1+y}{(1-x)^2} = \frac{2+2y}{(1-x)^2}$$

3. $y^2 = 10x$

$2y\frac{dy}{dx} = 10$, $\frac{dy}{dx} = \frac{10}{2y} = \frac{5}{y}$

$$\frac{d^2y}{dx^2} = \frac{y(0) - 5\frac{dy}{dx}}{y^2} = \frac{-5\left(\frac{5}{y}\right)}{y^2} = \frac{\frac{-25}{y}}{y^2} = \frac{-25}{y^3}$$