Chapter Three: Applications of Differentiation

3.1 Extrema on an Interval

Definition of Extrema – Let *f* be defined on an interval *I* containing *c*.

1. *f(c)* is the minimum of *f* on *I* if $f(c) \le f(x)$ for all *x* in *I*.

2. *f(c)* is the maximum of *f* on *I* if $f(c) \ge f(x)$ for all *x* in *I*.

The minimum and maximum of a function on an interval are the extreme values, or extrema, of the function on the interval. The minimum and maximum of a function on an interval are also called the absolute minimum and absolute maximum or global minimum and global maximum, on the interval.

The Extreme Value Theorem – If *f* is continuous on a closed interval [a, b], then *f* has both a minimum and a maximum on the interval.

Definition of Relative Extrema –

1. If there is an open interval containing *c* on which *f(c)* is a maximum, then *f(c)* is called a relative maximum of *f,* or you can say that *f* has a relative maximum at $(c, f(c))$.

2. If there is an open interval containing *c* on which *f(c)* is a minimum, then *f(c)* is called a relative minimum of *f,* or you can say that *f* has a relative minimum at $(c, f(c))$.

The plural of relative maximum is relative maxima, and the plural of relative minimum is relative minima. Relative maximum and relative minimum are sometimes called local maximum and local minimum, respectively.

Definition of a Critical Number – Let *f* be defined at *c.* If $f'(c) = 0$ or if *f* is not differentiable at *c*, then *c* is a critical number of *f.*

don't forget both possibilities

Theorem – If *f* has a relative minimum or relative maximum at *x = c*, then *c* is a critical number of *f*. That is, relative extrema occur only at critical numbers.

Guidelines for Finding Extrema on a Closed Interval – To find extrema of a continuous function *f* on a closed interval [a, b], use the following steps.

1. Find the critical numbers of *f* in (a, b). (x values where derivative is 0 or does not exist)

2. Evaluate *f* at each critical number in (a, b).

3. Evaluate *f* at each endpoint of [a, b].

4. The least of these values is the minimum. The greatest is the maximum.

Examples: Find the derivative of the function at the indicated extremum.

1. $f(x) = \cos\dfrac{\pi x}{2}$ at $(0,1)$ and $(2,-1)$

$f'(x) = -\sin\left(\frac{\pi x}{2}\right)\cdot\frac{\pi}{2} = -\frac{\pi}{2}\sin\left(\frac{\pi x}{2}\right)$

at $(0,1)$: $f'(0) = -\frac{\pi}{2}\sin\left(\frac{\pi(0)}{2}\right) = -\frac{\pi}{2}\sin(0) = -\frac{\pi}{2}(0) = 0$

at $(2,-1)$: $f'(2) = -\frac{\pi}{2}\sin\left(\frac{\pi(2)}{2}\right) = -\frac{\pi}{2}\sin(\pi) = -\frac{\pi}{2}(0) = 0$

Derivative is zero at extrema, no suprise here.

2. $f(x) = -3x\sqrt{x+1}$ at $(-1,0)$ and $\left(\dfrac{-2}{3}, \dfrac{2\sqrt{3}}{3}\right)$

$f'(x) = -3x\,\frac{1}{2}(x+1)^{-1/2}(1) + \sqrt{x+1}\,(-3)$

$f'(x) = \dfrac{-3x}{2\sqrt{x+1}} - 3\sqrt{x+1}$

at $(-1,0)$: $f'(-1) = \dfrac{-3(-1)}{2\sqrt{-1+1}} - 3\sqrt{-1+1}$ = undefined

at $\left(-\frac{2}{3}, \frac{2\sqrt{3}}{3}\right)$: $f'\left(-\frac{2}{3}\right) = \dfrac{-3\left(-\frac{2}{3}\right)}{2\sqrt{-\frac{2}{3}+1}} - 3\sqrt{-\frac{2}{3}+1}$

$= \dfrac{2}{2\sqrt{1/3}} - 3\sqrt{1/3}$

$= \sqrt{3} - \sqrt{3} = 0$

$\left\{ \dfrac{1}{\sqrt{\frac{1}{3}}} = \dfrac{1}{\frac{\sqrt{1}}{\sqrt{3}}} = \dfrac{\sqrt{3}}{\sqrt{1}} = \sqrt{3} \text{ and } 3\sqrt{\tfrac{1}{3}} = 3\dfrac{\sqrt{1}}{\sqrt{3}}\cdot\dfrac{\sqrt{3}}{\sqrt{3}} = 3\dfrac{\sqrt{3}}{3} = \sqrt{3} \right\} \longrightarrow$

Examples: Locate the absolute extrema of the function on the closed interval.

1. $f(x)=\dfrac{2x+5}{3},[0,5]$

$f(x)=\frac{2}{3}x+\frac{5}{3}$

$f'(x)=\frac{2}{3}$

f' is never zero and always defined

only possible extrema at endpoints

end point $f(0)=\frac{5}{3}$ minimum

end point $f(5)=5$ maximum

2. $f(x)=x^3-12x,[0,4]$ End points

$f'(x)=3x^2-12$

$0=3x^2-12$

$12=3x^2$

$4=x^2$

$\pm 2=x$

critical numbers

E.P. $f(0)=0$

C.N $f(2)=(2)^3-12(2)=8-24=-16$ min

E.P. $f(4)=(4)^3-12(4)=64-48=16$ max

C.N. $f(-2)$ not in interval

3. $g(x)=\sqrt[3]{x},[-1,1]$

$g(x)=x^{1/3}$

$g'(x)=\frac{1}{3}x^{-2/3}=\dfrac{1}{3\sqrt[3]{x^2}}$

$g'(x)$ is undefined at $x=0$ critical number

$g'(x)\neq 0$

end point $g(-1)=\sqrt[3]{-1}=-1$ minimum

critical number $g(0)=\sqrt[3]{0}=0$

end point $g(1)=\sqrt[3]{1}=1$ maximum

4. $h(t)=\dfrac{t}{t-2},[3,5]$

$h'(t)=\dfrac{(t-2)(1)-t(1)}{(t-2)^2}=\dfrac{t-2-t}{(t-2)^2}=\dfrac{-2}{(t-2)^2}$

$h'(t)$ undefined at $t=2$, not in interval

$h'(t)\neq 0$ no critical numbers

end point $h(3)=\dfrac{3}{3-2}=\dfrac{3}{1}=3$ maximum

end point $h(5)=\dfrac{5}{5-2}=\dfrac{5}{3}$ minimum

5. $g(x) = \sec x, \left[-\frac{\pi}{6}, \frac{\pi}{3}\right]$

$g'(x) = \sec x \tan x = \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x} = \frac{\sin x}{\cos^2 x}$

$g'(x)$ is undefined when $\cos x = 0$ so at $x = \frac{\pi}{2} + k\pi$ (not in interval)

$g'(x) = 0$ when $\sin x = 0$ so at $x = 0$ for this interval.
critical number

end point $g\left(-\frac{\pi}{6}\right) = \sec\left(-\frac{\pi}{6}\right) = \frac{2\sqrt{3}}{3} \approx 1.1547$

critical number $g(0) = \sec(0) = 1$ minimum

end point $g\left(\frac{\pi}{3}\right) = \sec\left(\frac{\pi}{3}\right) = 2$ maximum