3.2 Rolle's Theorem and the Mean Value Theorem

Rolle's Theorem – Let *f* be continuous on the closed interval [a, b] and differentiable on the open interval (a, b). If $f(a)=f(b)$ then there is at least one number *c* in (a, b) such that $f'(c)=0$.

Examples: Find the two x-intercepts of the function *f* and show that f'(x) = 0 at some point between the two x-intercepts.

1. $f(x)=x(x-3)$

x-intercepts when $0=x(x-3)$
so $x=0$ and $x=3$.

$f'(x)=x(1)+(x-3)(1)=x+x-3=2x-3$

$0=2x-3$
$3=2x$
$3/2=x$

$f'(\frac{3}{2})=0$ and $3/2$ is between x-intercepts of $x=0$ and $x=3$.

2. $f(x)=-3x\sqrt{x+1}$

x-intercepts at $x=0$ and $x=-1$

where $-3x=0$ where $\sqrt{x+1}=0$

$f'(x)=-3x\frac{1}{2}(x+1)^{-1/2}(1)-3\sqrt{x+1}$

$f'(x)=\frac{-3x}{2\sqrt{x+1}}-3\sqrt{x+1}$

$0=\frac{-3x}{2\sqrt{x+1}}-3\sqrt{x+1}$

$(2\sqrt{x+1})3\sqrt{x+1}=\frac{-3x}{2\cancel{\sqrt{x+1}}}\cdot\cancel{2\sqrt{x+1}}$

$6(x+1)=-3x$
$6x+6=-3x$
$6=-9x$
$-\frac{2}{3}=\frac{6}{-9}=x$

$-\frac{2}{3}$ is between $x=-1$ and $x=0$.

Examples: Determine whether Rolle's Theorem can be applied to *f* on the closed interval. If Rolle's Theorem can be applied, find all values *c* in the open interval such that f'(c) =0. If Rolle's Theorem cannot be applied, explain why not.

Check: Continuous on [a, b]
differentiable on (a, b)
$f(a)=f(b)$

1. $f(x)=x^2-5x+4,[1,4]$

✓ Continuous + diff. everywhere

✓ $f(1)=(1)^2-5(1)+4=1-5+4=0$
$f(4)=(4)^2-5(4)+4=16-20+4=0$

Rolle's Thm can be applied

$f'(x)=2x-5$
$f'(c)=0$ gives $0=2c-5$

$5=2c$
$\frac{5}{2}=c$

2. $f(x)=x^{2/3}-1,[-8,8]$ ✓Continuous

$f'(x) = \frac{2}{3}x^{-1/3} = \frac{2}{3\sqrt[3]{x}}$ Not differentiable at $x=0$
(derivative doesn't exist there)

Rolle's Theorem does not apply

3. $f(x)=\frac{x^2-2x-3}{x+2},[-1,3]$ not continuous or differentiable at $x=-2$, but that isn't in our interval.

$$\left.\begin{array}{l} f(-1) = \frac{(-1)^2-2(-1)-3}{-1+2} = \frac{1+2-3}{1} = 0 \\ f(3) = \frac{(3)^2-2(3)-3}{3+2} = \frac{9-6-3}{5} = 0 \end{array}\right\} ✓$$

Rolle's Theorem applies.

$$f'(x) = \frac{(x+2)(2x-2)-(x^2-2x-3)(1)}{(x+2)^2} = \frac{2x^2-2x+4x-4-x^2+2x+3}{(x+2)^2} = \frac{x^2+4x-1}{(x+2)^2}$$

If $f'(c)=0$ then $0 = \frac{x^2+4x-1}{(x+2)^2}$ or $0 = x^2+4x-1$

and $x = -2+\sqrt{5} \approx 0.236$

$x = -2-\sqrt{5} \approx -4.236$

we keep $c=-2+\sqrt{5}$
as it is the only one in the interval

4. $f(x)=\frac{x^2-1}{x},[-1,1]$

Not continuous at $x=0$, therefore Rolle's theorem does not apply.

5. $g(x)=\cos x,[0,2\pi]$

$$\left.\begin{array}{l} \text{Continuous and differentiable everywhere.} \\ g(0) = \cos 0 = 1 \\ g(2\pi) = \cos 2\pi = 1 \end{array}\right\} \text{Rolle's theorem applies.}$$

$g'(x) = -\sin x$

$0 = -\sin c$
when $c = 0, \pi, 2\pi$ all three are in the interval.

However, the theorem states that c is in the open interval so the only value we keep is $c=\pi$.

The Mean Value Theorem – If *f* is continuous on the closed interval [a, b] and differentiable on the open interval (a, b), then there exists a number *c* in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Examples: Determine whether the MVT can be applied to *f* on the closed interval. If the MVT can be applied, find all values of *c* given by the theorem. If the MVT cannot be applied, explain why not.

1. $f(x) = x^3, [0,1]$ continuous + differentiable everywhere, MVT applies

$f(1) = (1)^3 = 1$

$f(0) = (0)^3 = 0$

$f'(x) = 3x^2$

$f'(c) = 3c^2 = \frac{1-0}{1-0} = \frac{f(b)-f(a)}{b-a}$

$3c^2 = 1$

$c^2 = 1/3$ so $c = \pm\sqrt{\frac{1}{3}}$

Our answer is $c = \frac{\sqrt{3}}{3}$ after simplifying and considering the interval.

2. $f(x) = x^4 - 8x, [0,2]$ cont + diff ✓

note that Rolle's Thm also applies

$f(2) = (2)^4 - 8(2) = 16 - 16 = 0$

$f(0) = (0)^4 - 8(0) = 0$

$f'(x) = 4x^3 - 8$

$4c^3 - 8 = \frac{0-0}{2\ 0}$

$4c^3 - 8 = 0$

$4c^3 = 8$

$c^3 = 2$

$c = \sqrt[3]{2}$

3. $f(x) = \frac{x+1}{x}, [-1,2]$ Not continuous on $[-1,2]$. MVT does not apply

4. $f(x) = \sin x, [0,\pi]$ cont. + differentiable ✓

$f(\pi) = \sin \pi = 0$

$f(0) = \sin 0 = 0$

$f'(x) = \cos x$

$\cos c = \frac{0-0}{\pi - 0}$

$\cos c = 0$

$c = \frac{\pi}{2}$

Rolle's theorem is a special case of the Mean Value Theorem in which the endpoints have the same value.

Example: A plane begins its takeoff at 2:00 PM on a 2500 mile flight. After 5.5 hours, the plane arrives at its destination. Explain why there are at least two times during the flight when the speed of the plane is 400 mph.

On average, the plane flew at $\frac{2500}{5.5} \frac{\text{miles}}{\text{hr}} \approx 454.5$ mph

In order to average this speed it had to go from 0 mph up to full speed, past 454.5 mph, and then it had to power back down to land. This is an application of the MVT.