3.3 Increasing and Decreasing Functions and the First Derivative Test

Definitions of Increasing and Decreasing Functions –

A function *f* is increasing on an interval if for any two numbers x_1 and x_2 in the interval, $x_1 < x_2$ implies $f(x_1) < f(x_2)$.

A function *f* is decreasing on an interval if for any two numbers x_1 and x_2 in the interval, $x_1 < x_2$ implies $f(x_1) > f(x_2)$.

Theorem 3.5 – Let *f* be a function that is continuous on the closed interval [a, b] and differentiable on the open interval (a, b).

1. If $f'(x) > 0$ for all *x* in (a, b), then *f* is increasing on [a, b].

2. If $f'(x) < 0$ for all *x* in (a, b), then *f* is decreasing on [a, b].

3. If $f'(x) = 0$ for all *x* in (a, b), then *f* is constant on [a, b].

Guidelines for Finding Intervals on which a Function is Increasing or Decreasing – Let *f* be continuous on the interval (a, b). To find the open intervals on which *f* is increasing or decreasing, use the following steps.

1. Locate the critical numbers of *f* in (a, b), and use these numbers to determine test intervals.

2. Determine the sign of f'(x) at one test value in each of the intervals.

3. Use Theorem 3.5 to determine whether *f* is increasing or decreasing on each interval.

Examples: Identify the open intervals on which the function is increasing or decreasing.

1. $h(x) = 27x - x^3$

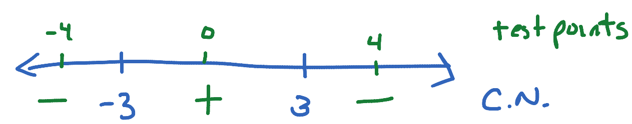
$h'(x) = 27 - 3x^2$

$0 = 27 - 3x^2$

$3x^2 = 27$

$x^2 = 9$

$x = \pm 3$ critical numbers



$f'(-4) = 27 - 3(-4)^2 = 27 - 3(16) =$ Negative

$f'(0) = 27 - 3(0)^2 =$ positive

$f'(4) = 27 - 3(4)^2 =$ negative

increasing $(-3, 3)$

decreasing $(-\infty, -3) \cup (3, \infty)$

2. $y = x + \frac{4}{x}$

$y' = 1 - \frac{4}{x^2}$

$x = 0$ is a critical number

$0 = 1 - \frac{4}{x^2}$

$\frac{4}{x^2} = 1 \Rightarrow x^2 = 4$

$x = \pm 2$ critical numbers

-3 -1 1 3 test points.

+ -2 − 0 − 2 + critical numbers

$y'(-3) = 1 - \frac{4}{(-3)^2} = 1 - \frac{1}{9} =$ positive $= y'(3)$

$y'(-1) = 1 - \frac{4}{(-1)^2} = 1 - 4 =$ negative $= y'(1)$

increasing $(-\infty, -2) \cup (2, \infty)$

decreasing $(-2, 0) \cup (0, 2)$

Note that we must exclude $x = 0$ as the function is not defined there.

The First Derivative Test – Let *c* be a critical number of a function *f* that is continuous on an open interval *I* containing *c*. If *f* is differentiable on the interval, except possibly at *c*, then f(c) can be classified as follows.

1. If f′(x) changes from negative to positive at *c*, then *f* has a relative minimum at (c, f(c)).

2. If f′(x) changes from positive to negative at *c*, then *f* has a relative maximum at (c, f(c)).

3. If f′(x) is positive on both sides of *c* or negative on both sides of *c*, then f(c) is neither a relative minimum nor a relative maximum.

Examples: (a) Find the critical numbers of *f*, (b) find the open intervals on which the function is increasing or decreasing, and (c) apply the First Derivative Test to identify all relative extrema.

1. $f(x) = x^2 + 6x + 10$

$f'(x) = 2x + 6$

$0 = 2x + 6$

$-6 = 2x$

(a) $-3 = x$ critical number

-4 0 test points

− -3 +

$f'(-4) = 2(-4) + 6 =$ negative

$f'(0) = 2(0) + 6 =$ positive

(b) increasing $(-3, \infty)$

decreasing $(-\infty, -3)$

(c) Changes decreasing (−) to increasing (+) at $x = -3$, so a local minimum occurs at $(-3, f(-3)) = (-3, 1)$.

2. $f(x) = x^3 - 6x^2 + 15$

$f'(x) = 3x^2 - 12x$

$0 = 3x^2 - 12x$

$0 = 3x(x-4)$

(a) $x=0, x=4$ critical numbers

-1   1   5   test points

\+   0   −   4   \+

$f'(-1) = 3(-1)(-1-4) = - \cdot - =$ positive

$f'(1) = 3(1)(1-4) = + \cdot - =$ negative

$f'(5) = 3(5)(5-4) = + \cdot + =$ positive

all that matters is sign, not value. Test each factor for + or −.

(b) increasing $(-\infty, 0) \cup (4, \infty)$

decreasing $(0, 4)$

(c) + to − at $x=0 \Rightarrow$ maximum at $(0, 15)$

− to + at $x=4 \Rightarrow$ minimum at $(4, -17)$

3. $f(x) = x^4 - 32x + 4$

$f'(x) = 4x^3 - 32$

$0 = 4x^3 - 32$

$32 = 4x^3$

$8 = x^3$

(a) $2 = x$ C.N.

0   3   test

−   2   \+

$f'(0) = -32 =$ negative

$f'(3) = 4(3)^3 - 32 =$ positive

(b) increasing $(2, \infty)$

decreasing $(-\infty, 2)$

(c) local minimum at $(2, -44)$

Fact: The derivative tells us where to find the min/max, but the original function tells us what it is.

4. $f(x) = (x-3)^{1/3}$

$f'(x) = \frac{1}{3}(x-3)^{-2/3}(1)$

or $f'(x) = \dfrac{1}{\sqrt[3]{(x-3)^2}}$

$f'(x) \neq 0$

$f'(x)$ is undefined at $x=3$

(a) critical number $x=3$

0   5

\+   3   \+

$f'(0) = \dfrac{1}{\sqrt[3]{(0-3)^2}} =$ pos

$f'(5) = \dfrac{1}{\sqrt[3]{(5-3)^2}} -$ pos

(b) increasing $(-\infty, 3) \cup (3, \infty)$

(c) no change in sign of derivative so no max/min.

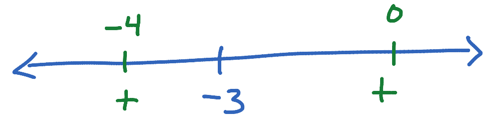
Even though the graph appears to be always increasing, it is not increasing at $x=3$. This is the location of a vertical tangent line

5. $f(x)=\dfrac{x}{x+3}$

$$f'(x)=\frac{(x+3)(1)-x(1)}{(x+3)^2}$$

$$f'(x)=\frac{x+3-x}{(x+3)^2}=\frac{3}{(x+3)^2}$$

(a) Critical number $x=-3$
(f' not def.)



$f'(-4)=\dfrac{3}{(-4+3)^2}=\dfrac{pos}{pos}=$ positive

$f'(0)=\dfrac{3}{pos}=$ positive

(b) increasing $(-\infty,-3)\cup(-3,\infty)$

(c) no change of sign in f', no extrema

6. $f(x)=\dfrac{x^2-3x-4}{x-2}$

$$f'(x)=\frac{(x-2)(2x-3)-(x^2-3x-4)(1)}{(x-2)^2}$$

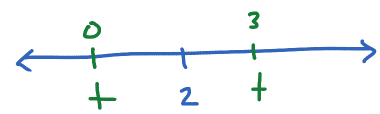
$$f'(x)=\frac{2x^2-3x-4x+6-x^2+3x+4}{(x-2)^2}$$

$$f'(x)=\frac{x^2-4x+10}{(x-2)^2}$$

$\leftarrow$ no real zeros

$\leftarrow$ $x=2$ undefined

(a) $x=2$ critical number



$f'(0)=\dfrac{10}{pos}=$ positive

$f'(3)=\dfrac{7}{pos}=$ positive

(b) increasing $(-\infty,2)\cup(2\ \infty)$

(c) no extrema

7. $f(x)=\sin x\cos x+5,(0,2\pi)$

$$f'(x)=\sin x(-\sin x)+\cos x(\cos x)+0$$

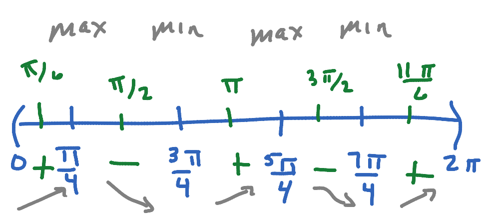
$$f'(x)=\cos^2 x-\sin^2 x$$

$$0=\cos^2 x-\sin^2 x$$

$$\sin^2 x=\cos^2 x$$

at $x=\frac{\pi}{4}$ in all 4 quadrants

(a) $x=\frac{\pi}{4},\frac{3\pi}{4},\frac{5\pi}{4},\frac{7\pi}{4}$



$f'(\frac{\pi}{6})=\frac{1}{2}$ $\qquad$ $f'(\frac{\pi}{2})=-1$

$f'(\pi)=1$ $\qquad$ $f'(\frac{3\pi}{2})=-1$

$f(\frac{11\pi}{6})=\frac{1}{2}$

(b) increasing $(0,\pi/4)\cup(\frac{3\pi}{4},\frac{5\pi}{4})\cup(\frac{7\pi}{4},2\pi)$

decreasing $(\pi/4,3\pi/4)\cup(5\pi/4,\frac{7\pi}{4})$

(c) Maximums at $(\pi/4,\frac{11}{2})$ and $(\frac{5\pi}{4},\frac{11}{2})$
Minimums at $(\frac{3\pi}{4},\frac{9}{2})$ and $(\frac{7\pi}{4},\frac{9}{2})$
$\Big\}$ points $(x\ y)$

intervals (x_1,x_2)

Example: Coughing forces the trachea (windpipe) to contract, which affects the velocity *v* of the air passing through the trachea. The velocity of the air during coughing is $v = k(R-r)r^2,\ 0 \le r < R$, where *k* is a constant, *R* is the normal radius of the trachea, and *r* is the radius during coughing. What radius will produce the maximum air velocity?

Find r during cough

derivative = 0

$$\frac{dv}{dr} = k\left[(R-r)2r + r^2(-1)\right]$$
$$= k\left[2Rr - 2r^2 \quad r^2\right]$$
$$= k\left[2Rr - 3r^2\right]$$

$\frac{dv}{dr} = 0$ when $0 = k\left[2Rr - 3r^2\right]$

$0 = kr(2R - 3r)$

$kr = 0,\ r = 0$, radius $= 0$ is not a max velocity as no air can flow

$2R - 3r = 0$

$2R = 3r$

$\frac{2R}{3} = r$, radius that is $\frac{2}{3}$ the normal radius of your trachea will provide max velocity of a cough.

Good luck making that happen on purpose!