3.4 Concavity and the Second Derivative Test

Definition of Concavity – Let *f* be differentiable on an open interval *I*. The graph of *f* is concave upward on *I* if f' is increasing on the interval and concave downward on *I* if f' is decreasing on the interval.

Notice this is talking about the derivative increasing or decreasing.

For the derivative to be increasing or decreasing we need to look at its derivative. That is, to determine if f' is increasing or decreasing we have to find f''.

Test for Concavity – Let *f* be a function whose second derivative exists on an open interval *I*.

1. If $f''(x) > 0$ for all *x* in *I*, then the graph of *f* is concave upward on *I*.
2. If $f''(x) < 0$ for all *x* in *I*, then the graph of *f* is concave downward on *I*.

Basically, this means we need to do a sign test for intervals of the second derivative as well.

Definition of Point of Inflection – Let *f* be a function that is continuous on an open interval and let *c* be a point in the interval. If the graph of *f* has a tangent line at this point (c, f(c)), then this point is a point of inflection of the graph of *f* if the concavity of *f* changes from upward to downward (or downward to upward) at the point.

Theorem 3.8 – If (c, f(c)) is a point of inflection of the graph of *f*, then either $f''(x) = 0$ or *f''* does not exist at x = c. Similar to critical numbers and the first derivative.

Examples: Determine the open intervals on which the graph is concave upward or concave downward.

f''

1. $f(x) = -x^3 + 3x^2 - 2$

$f'(x) = -3x^2 + 6x$

$f''(x) = -6x + 6$

$0 = -6x + 6$

$6x = 6$

$x = 1$

$+ \quad -$

$0 \quad 1 \quad 2$

$f''(0) = -6(0) + 6 = 6$ positive $\Rightarrow$ concave up

$f''(2) = -6(2) + 6 = -6$ negative $\Rightarrow$ concave down

cc up $(-\infty, 0)$ cc down $(0, \infty)$

2. $y = x^5 - 5x + 2$

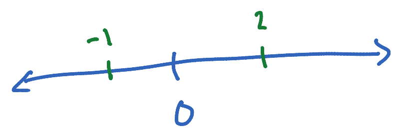
$y' = 5x^4 - 5$

$y'' = 20x^3$

$y'' = 0$ when $x = 0$

for $x < 0$, $y'' < 0$ concave down.
for $x > 0$, $y'' > 0$ concave up.

If you don't recognize this, work it out.



$f''(-1) = 20(-1)^3 = -20 = \text{neg}$

$f''(2) = 20(2)^3 = 160 = \text{pos}.$

cc up $(0, \infty)$ cc down $(-\infty, 0)$

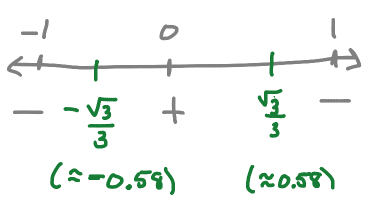
3. $f(x) = \dfrac{x^2}{x^2+1}$

$$f'(x) = \frac{(x^2+1)(2x) - x^2(2x)}{(x^2+1)^2} = \frac{2x^3 + 2x - 2x^3}{(x^2+1)^2} = \frac{2x}{(x^2+1)^2}$$

$$f''(x) = \frac{(x^2+1)^{\cancel{2}}(2) - 2x \cdot 2(\cancel{x^2+1})(2x)}{(x^2+1)^{\cancel{4}\,3}} = \frac{2(x^2+1) - 8x^2}{(x^2+1)^3} = \frac{2-6x^2}{(x^2+1)^3}$$

denom $= (x^2+1)^3$ is always positive because $x^2 \geq 0$ so $x^2+1 > 0$ and a positive cubed is still positive.

numerator $= 2 - 6x^2 = 0 \Rightarrow 2 = 6x^2 \Rightarrow \frac{1}{3} = x^2 \Rightarrow \pm\sqrt{\frac{1}{3}} = x$



$f''(-1) = \dfrac{2-6(-1)^2}{\text{pos}} = \dfrac{-4}{\text{pos}} = \text{neg}$

$f''(0) = \dfrac{2}{\text{pos}} = \text{pos}$

$f''(1) = \dfrac{-4}{\text{pos}} = \text{neg}$

cc up $\left(-\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right)$

cc down $\left(-\infty, -\frac{\sqrt{3}}{3}\right) \cup \left(\frac{\sqrt{3}}{3}, \infty\right)$ ← Exact answers only. Decimals just give us an idea of useful test points.

Examples: Find the points of inflection and discuss the concavity of the graph of the function.

1. $f(x) = -x^4 + 24x^2$

$f'(x) = -4x^3 + 48x$

$f''(x) = -12x^2 + 48$

$0 = -12x^2 + 48$

$12x^2 = 48$

$x^2 = 4$

$x = \pm 2$

$-4 \quad 0 \quad 3$

$- \quad -2 \quad + \quad 2 \quad - \qquad f''$

$f''(-4) =$ negative

$f''(0) =$ positive

$f''(3) =$ negative

Concavity: up $(-2,2)$; down $(-\infty,-2) \cup (2,\infty)$

points of inflection occur at $x=-2$ and $x=2$. To find points always use the original function. $(-2, f(-2)) = (-2, 80)$ and $(2, f(2)) = (2, 80)$

2. $f(x) = x\sqrt{9-x}$

$f'(x) = x\frac{1}{2}(9-x)^{-1/2}(-1) + \sqrt{9-x}$

simplify $f'(x) = \dfrac{-x}{2\sqrt{9-x}} + \sqrt{9-x}$

$$f''(x) = \frac{2\sqrt{9-x}(-1) + x\,2\,\frac{1}{2}(9-x)^{-1/2}(-1)}{\left(2\sqrt{9-x}\right)^2} + \frac{1}{2}(9-x)^{-1/2}(-1)$$

Simplify

$$f''(x) = \frac{-2\sqrt{9-x} - \frac{x}{\sqrt{9-x}}}{4(9-x)} - \frac{1}{2\sqrt{9-x}} = \frac{\frac{-2(9-x)}{\sqrt{9-x}} - \frac{x}{\sqrt{9-x}}}{4(9-x)} - \frac{1}{2\sqrt{9-x}}$$

$$f''(x) = \frac{-18+2x-x}{4(9-x)^{3/2}} - \frac{1}{2\sqrt{9-x}} = \frac{-18+x}{4(9-x)^{3/2}} - \frac{1}{2\sqrt{9-x}} \cdot \frac{2(9-x)}{2(9-x)} = \frac{-18+x-18+x}{4(9-x)^{3/2}} = \frac{2x-36}{4(9-x)^{3/2}}$$

$$f''(x) = \frac{2(x-18)}{2(2(9-x)^{3/2})} = \frac{x-18}{2(9-x)^{3/2}}$$

whew! That was rough!

f'' is undefined at $x=9$ and f'' is zero at $x=18$

However, the original domain is $9-x \geq 0 \rightarrow -x \geq -9 \rightarrow x \leq +9$

For any $x \leq 9$, $f''(x) = \dfrac{neg}{pos} = neg$ so f is concave down on its domain $(-\infty, +9)$ with no inflection points

Second Derivative Test – Let *f* be a function such that f'(c) = 0 and the second derivative of *f* exists on an open interval containing *c*.

1. If $f''(c)>0$, then *f* has a relative minimum at $(c, f(c))$.

2. If $f''(c)<0$, then *f* has a relative maximum at $(c, f(c))$.

If $f''(c)=0$, the test fails. That is, *f* may have a relative maximum, a relative minimum, or neither. In such cases, you can use the First Derivative Test.

Examples: Find all relative extrema. Use the Second Derivative Test where applicable.

1. $f(x)=-(x-5)^2$

max/min/inc/dec $f'(x) = -2(x-5)(1) = -2x+10$

p.o.i/concavity $f''(x) = -2$

$f'(x)=0 \Rightarrow 0=-2x+10$ so $x=5$

test points: 0, 6

$+$ | 5 | $-$

$f'(0)=10$ $\quad f'(6)=-2$

Increasing $(-\infty,5)$ decreasing $(5,\infty)$

Max at $(5, f(5)) = (5,0)$

$f''(x)=0$ never so no point of inflection

$f''(x)<0$ always so concave down on $(-\infty,\infty)$. → second derivative negative confirms maximum

2. $f(x)=x^3-5x^2+7x$

$f'(x) = 3x^2-10x+7$ → $f''(x) = 6x-10$

$f'(x) = (3x-7)(x-1)$

$f'(x)=0$ when $x = \frac{7}{3}, 1$

$f''(x)=0$ when $0=6x-10$

$10=6x$

$\frac{10}{6} = x = \frac{5}{3}$

0 2 3

$+$ | 1 | $-$ | $\frac{7}{3}$ | $+$

0 5

$-$ | 1 | $\frac{5}{3}$ | $\frac{7}{3}$ | $+$

$f''(0)=-10$ $\quad f''(5)=20$

$f'(0)=(-7)(-1)=$ positive

$f'(2)=(-1)(1)=$ negative

$f'(3)=(2)(2)=$ positive

inc $(-\infty,1)\cup(\frac{7}{3},\infty)$

dec $(1,\frac{7}{3})$

max at $(1,3)$

min at $(\frac{7}{3},\frac{49}{27})$

$f''(\frac{7}{3})=$ pos so a min at $x=\frac{7}{3}$

$f''(1)=$ neg so a max at $x=1$

P.O.I. at $(5/3, 65/27)$ cc up $(5/3,\infty)$ cc down $(-\infty, 5/3)$

3. $g(x) = -\frac{1}{8}(x+2)^2(x-4)^2$

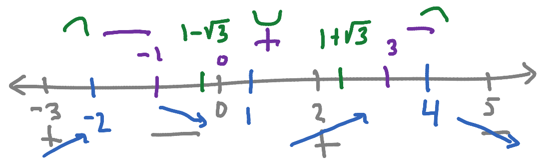
$$g'(x) = -\frac{1}{8}\left[(x+2)^2 2(x-4)(1) + (x-4)^2 2(x+2)\right] = -\frac{1}{8}\left[2(x+2)(x-4)(x+2+x-4)\right]$$

$$g'(x) = -\frac{1}{4}(x+2)(x-4)(2x-2) = -\frac{1}{2}(x+2)(x-4)(x-1)$$ Zeros at $x = -2, 4, 1$

Simplify: $g'(x) = -\frac{1}{2}(x^2-2x-8)(x-1)$

$$g''(x) = -\frac{1}{2}\left[(x^2-2x-8)(1) + (x-1)(2x-2)\right] = -\frac{1}{2}\left[x^2-2x-8+2x^2-2x-2x+2\right]$$

$$g''(x) = -\frac{1}{2}(3x^2-6x-6) = -\frac{3}{2}(x^2-2x-2)$$ Zeros at $x = 1 \pm \sqrt{3}$ (quad formula) (approx 2.7, -.7)



$g'(-3) = (-)(-)(-)(-) = +$
$g'(0) = (-)(+)(-)(-) = -$
$g'(2) = (-)(+)(-)(+) = +$
$g'(5) = (-)(+)(+)(+) = -$

Inc $(-\infty,-2) \cup (1,4)$
dec $(-2,1) \cup (4,\infty)$
max at $(-2,0)$ & $(4,0)$
min at $(1,-\frac{81}{8})$

$g''(-1) = -\frac{3}{2}(1) =$ neg
$g''(0) = -\frac{3}{2}(-2) =$ pos
$g''(3) = -\frac{3}{2}(1) =$ neg

p.o.inflection $(1-\sqrt{3}, -\frac{9}{2})$ and $(1+\sqrt{3}, -\frac{9}{2})$

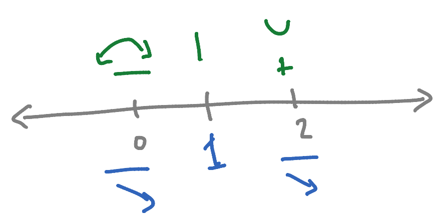
cc up $(1-\sqrt{3}, 1+\sqrt{3})$
cc down $(-\infty, 1-\sqrt{3}) \cup (1+\sqrt{3}, \infty)$

4. $y = \frac{x}{x-1}$

$$y' = \frac{(x-1)(1) - x(1)}{(x-1)^2} = \frac{-1}{(x-1)^2} = -(x-1)^{-2}$$

$$y'' = 2(x-1)^{-3} = \frac{2}{(x-1)^3}$$

Both undefined at $x=1$, as is the original function. No other critical points.



$y'(0) = \frac{-1}{(0-1)^2} = \frac{-1}{pos} =$ neg
$y'(2) = \frac{-1}{pos} =$ neg

dec $(-\infty,1) \cup (1,\infty)$
no max/min

$y''(0) = \frac{2}{(0-1)^3} = \frac{2}{-1} =$ neg
$y''(2) = \frac{2}{(2-1)^3} = \frac{2}{1} =$ pos

cc down $(-\infty,1)$
cc up $(1,\infty)$
No point of inflection as $x=1$ is not in the domain.