3.5 Limits at Infinity

Definition of Limits at Infinity – Let L be a real number.

1. The statement $\lim_{x\to\infty} f(x) = L$ means that for each ε > 0 there exists an $M > 0$ such that $|f(x) - L| < \varepsilon$ whenever $x > M$.

2. The statement $\lim_{x\to-\infty} f(x) = L$ means that for each ε > 0 there exists an $N < 0$ such that $|f(x) - L| < \varepsilon$ whenever $x < N$.

Definition of a Horizontal Asymptote – The line $y = L$ is a horizontal asymptote of the graph of f if

$$\lim_{x\to\infty} f(x) = L \ \underline{or}\ \lim_{x\to-\infty} f(x) = L.$$

one or the other or both

Theorem 3.10 (Limits at Infinity) – If r is a positive rational number and c is any real number, then $\lim_{x\to\infty} \frac{c}{x^r} = 0$. Furthermore, if x^r is defined when $x < 0$, then $\lim_{x\to-\infty} \frac{c}{x^r} = 0$.

This will be used frequently!

Guidelines for Finding Limits at $\pm\infty$ of Rational Functions –

1. If the degree of the numerator is less than the degree of the denominator, then the limit of the rational function is 0. (That is, the horizontal asymptote is $y = 0$.)

2. If the degree of the numerator is equal to the degree of the denominator, then the limit of the rational function is the ratio of the leading coefficients.

3. If the degree of the numerator is greater than the degree of the denominator, then the limit of the rational function does not exist.

Examples: Find the limit

1. $\lim_{x\to-\infty}\left(\frac{5}{x} - \frac{x}{3}\right) = \lim_{x\to-\infty} \frac{15 - x^2}{3x}$ = dne because the degree of the numerator is larger than the degree of the denominator

$\frac{3}{3} \cdot \frac{5}{x} - \frac{x}{3} \cdot \frac{x}{x} = \frac{15 - x^2}{3x}$

2. $\lim_{x\to\infty} \frac{x^2+3}{2x^2-1} = \lim_{x\to\infty} \frac{\frac{x^2}{x^2}+\frac{3}{x^2}}{\frac{2x^2}{x^2}-\frac{1}{x^2}} = \lim_{x\to\infty} \frac{1+\cancel{\frac{3}{x^2}}^{\,0}}{2-\cancel{\frac{1}{x^2}}^{\,0}} = \frac{1}{2}$

Divide each term by the highest power of x in entire expression

Simplify each little fraction

use theorem 3.10: $\lim_{x\to\infty} \frac{c}{x^r} = 0$

3. $\lim_{x\to-\infty} \frac{x}{\sqrt{x^2+1}} = \lim_{x\to-\infty} \frac{\frac{x}{x}}{\sqrt{\frac{x^2}{x^2}+\frac{1}{x^2}}} = \lim_{x\to-\infty} \frac{1}{\sqrt{1+\frac{1}{x^2}}} = \frac{1}{\sqrt{1+0}} = 1$

Notice that $x = \sqrt{x^2}$ so inside the radical divide by x^2 and outside the radical divide by x

simplify

theorem 3.10

4. $\lim_{x\to\infty} \cos\frac{1}{x} = \cos(0) = 1$

as $x\to\infty$, $\frac{1}{x}\to 0$ by theorem 3.10

Examples: Sketch the graph of the equation using extrema, intercepts, symmetry, and asymptotes.

1. $y = \frac{x-4}{x-3}$

from $y = \frac{x-4}{x-3}$ we find

x-int at $(4,0)$

y-int at $(0,\frac{-4}{-3}) = (0,\frac{4}{3})$

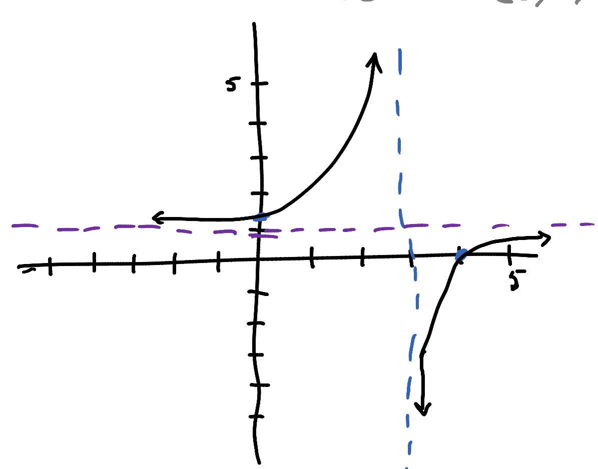
Vert. asym $x=3$

from $y' = \frac{(x-3)(1)-(x-4)(1)}{(x-3)^2}$

$y' = \frac{x-3-x+4}{(x-3)^2} = \frac{1}{(x-3)^2}$

y' always positive so increasing $(-\infty,3)\cup(3,\infty)$ no max/min

from $y'' = \frac{-2}{(x-3)^3}$ we find no points of inflection. cc up $(-\infty,3)$ cc down $(3,\infty)$



$\lim_{x\to\infty} y = 1$ and $\lim_{x\to-\infty} y = 1$ so $y=1$ is a Horiz. asymptote

2. $y = \dfrac{3x}{1-x^2}$

$y = \dfrac{3x}{1-x^2}$

x-int $(0,0)$

y-int $(0,0)$

Vert. asy. $x=1, x=-1$

notice equations

$\lim_{x\to\infty} \dfrac{3x}{1-x^2} = 0$

so horiz. asy $y=0$

$y' = \dfrac{(1-x^2)(3) - 3x(-2x)}{(1-x^2)^2}$

$y' = \dfrac{3-3x^2+6x^2}{(1-x^2)^2}$

$y' = \dfrac{3+3x^2}{(1-x^2)^2} = \dfrac{3(1+x^2)}{(1-x^2)^2}$

$y' \neq 0$ no max/min

numerator & denominator always positive so increasing $(-\infty,-1)\cup(-1,1)\cup(1,\infty)$

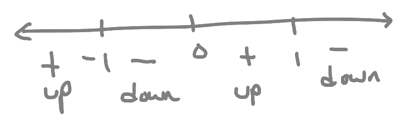
$y'' = \dfrac{(1-x^2)^{\cancel{2}\,1}(6x) - (3+3x^2)2(\cancel{1-x^2})(-2x)}{(1-x^2)^{\cancel{4}\,3}}$

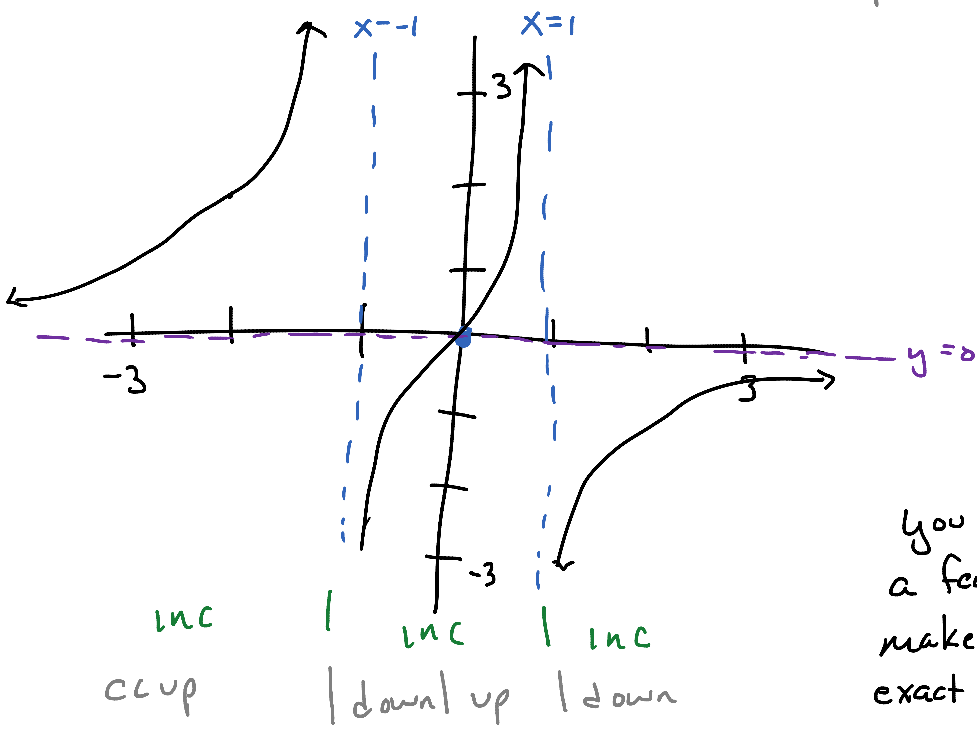
$y'' = \dfrac{(1-x^2)(6x) + 4x(3+3x^2)}{(1-x^2)^3}$

$y'' = \dfrac{6x - 6x^3 + 12x + 12x^3}{(1-x^2)^3}$

$y'' = \dfrac{6x^3+18x}{(1-x^2)^3} = \dfrac{6x(x^2+3)}{(1-x^2)^3}$

$y'' = 0$ at $x=0$





you should find a few points to make sure of the exact graph