3.7 Optimization Problems

Guidelines for Solving Applied Minimum and Maximum Problems

1. Identify all given quantities and all quantities to be determined. If possible, make a sketch.

2. Write a preliminary equation for the quantity that is to be maximized or minimized.

3. Reduce the primary equation to one having a single independent variable. This may involve the use of secondary equations relating the independent variables of the primary equation.

4. Determine the feasible domain of the primary equation. That is, determine the values for which the stated problem makes sense.

5. Determine the desired maximum or minimum value by the calculus techniques of chapter 3.

Example 1: Find two positive numbers such that the product is 185 and the sum is a minimum.

$x+y$ multiply add derivative

$xy = 185$

$S = x + y$ objective function

if $xy = 185$, $y = \frac{185}{x}$. Substitute into the sum equation to eliminate a variable

$S = x + \frac{185}{x} = x + 185x^{-1}$

$S' = 1 - 185x^{-2} = 1 - \frac{185}{x^2}$

Minimum occurs where $S' = 0$ so solve

$0 = 1 - \frac{185}{x^2}$

$\frac{185}{x^2} = 1$ so $185 = x^2$ and $x = \sqrt{185}$ (positive only)

If $x = \sqrt{185}$, $y = \frac{185}{\sqrt{185}} = \sqrt{185}$ also

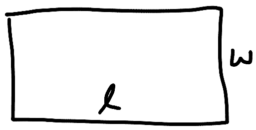
exact, not decimal

Check: Is $x = \sqrt{185}$ a minimum?

$S'' = +2(185)x^{-3} = \frac{370}{x^3}$

$S''(\sqrt{185}) = \frac{370}{(\sqrt{185})^3}$ = positive so is a minimum

Example 2: Find the length and width of a rectangle that has a perimeter of 80 meters and a maximum area.



$P = 80 = 2l + 2w$

or $40 = l + w$

$40 - w = l$

$A = lw$

$A = (40 - w)w = 40w - w^2$

$A' = 40 - 2w$

$0 = 40 - 2w$

$20 = w$

$A'' = -2$ so is always cc down and 20 is a max.

If $w = 20m$ then $l = 20m$

Note: A rectangle of max area will always be a square.

Example 3: Find the point on the graph of the function $f(x) = (x-1)^2$ that is closest to the point (-5,3).

$y = (x-1)^2$

distance $= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

minimize

$d = \sqrt{(-5-x)^2 + (3-(x-1)^2)^2}$

This derivative will be much easier if we simplify the radicand to polynomial form.

$d = \sqrt{x^4 - 4x^3 + x^2 + 18x + 29}$

$d' = \frac{1}{2}(x^4 - 4x^3 + x^2 + 18x + 29)^{-1/2}(4x^3 - 12x^2 + 2x + 18) = \dfrac{4x^3 - 12x^2 + 2x + 18}{2\sqrt{x^4 - 4x^3 + x^2 + 18x + 29}}$

or $d' = \dfrac{2x^3 - 6x^2 + x + 9}{\sqrt{x^4 - 4x^3 + x^2 + 18x + 29}}$

We could use the second derivative test for minimum, but that's not looking fun. Let's just trust our algebra on this one!

$d' = 0$ when numerator is zero

Using techiques from Pre-Calculus this happens at $x = -1$. (The other two roots are complex.)

When $x = -1$, $y = (-1-1)^2 = (-2)^2 = 4$

The point $(-1, 4)$ on $f(x) = (x-1)^2$ is Closest to $(-5\ 3)$.

Example 4: On a given day, the flow rate F (cars per hour) on a congested roadway is $F = \dfrac{v}{22 + 0.02v^2}$ where v is the speed of the traffic in miles per hour. What speed will maximize the flow rate on the road?

$F' = 0$

$$F' = \frac{(22+0.02v^2)(1) - v(0.04v)}{(22+0.02v^2)^2} = \frac{22 - 0.02v^2}{(22+0.02v^2)^2}$$

$F' = 0$ when $22 - 0.02v^2 = 0$

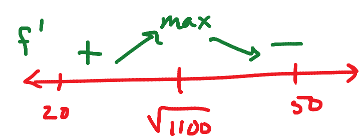
$22 = 0.02v^2$

$1100 = v^2$

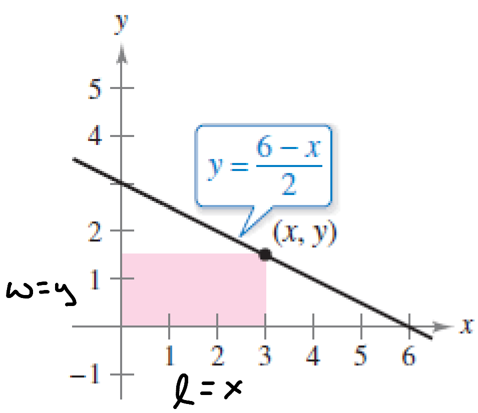
$33 \text{ mph} \approx \sqrt{1100} = v$

Notice F' is never undefined.

How do we know this is a max and not a minimum?



Example 5: A rectangle is bounded by the x- and y-axes and the graph of $y = \dfrac{6-x}{2}$. What length and width should the rectangle have so that its area is a maximum?



$A = xy$ with $y = \frac{6-x}{2}$ gives $A = x\left(\frac{6-x}{2}\right)$

or $A = \frac{6x - x^2}{2} = 3x - \frac{1}{2}x^2$

maximum at $A' = 0$

$A' = 3 - \frac{1}{2}(2x)$ or $A' = 3 - x$ and $A' = 0$ when $x = 3$.

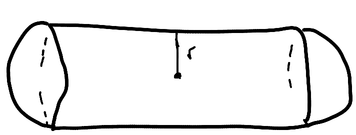
Rectangle length is 3 units and width is

$y = \frac{6-3}{2} = \frac{3}{2}$ units

Do not count on accuracy of figures in general!

Example 6: A solid is formed by adjoining two hemispheres to the ends of a right circular cylinder. The total volume of the solid is 14 cubic centimeters. Find the radius of the cylinder that produces the minimum surface area.

two hemispheres = 1 sphere $V = \frac{4}{3}\pi r^3 + \pi r^2 h = 14$



minimize $SA = 4\pi r^2 + 2\pi r h$

we must solve for h in Volume to sub. in Surface area

$$h = \frac{14 - \frac{4}{3}\pi r^3}{\pi r^2}$$

$$SA = 4\pi r^2 + 2\cancel{\pi r}\left(\frac{14 - \frac{4}{3}\pi r^3}{\cancel{\pi} r^{\cancel{2}}}\right)$$

$$SA = 4\pi r^2 + \frac{28}{r} - \frac{8}{3}\pi r^2$$

$$SA = \frac{4}{3}\pi r^2 + \frac{28}{r}$$

$$(SA)' = \frac{8\pi}{3} r - \frac{28}{r^2}$$

$$0 = \frac{8\pi}{3} r - \frac{28}{r^2}$$

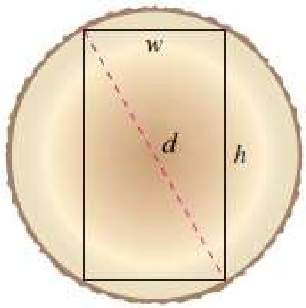
$$\frac{28}{r^2} = \frac{8\pi r}{3}$$

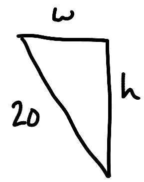
$$84 = 8\pi r^3$$

$$r^3 = \frac{84}{8\pi}$$

$$r = \sqrt[3]{\frac{84}{8\pi}} \approx 1.495 \text{ cm}$$

Example 7: A wooden beam has a rectangular cross section of height *h* and width *w*. The strength *S* of the beam is directly proportional to the width and the square of the height by $S = kwh^2$, where *k* is the proportionality constant. What are the dimensions of the strongest beam that can be cut from a round log of diameter 20 inches?





$w^2 + h^2 = 20^2$ or $h^2 = 400 - w^2$

$S = kwh^2 = kw(400 - w^2) = 400kw - kw^3$

$S' = 400k - 3kw^2$

$0 = 400k - 3kw^2$

$3kw^2 = 400k$

$w^2 = \frac{400}{3}$

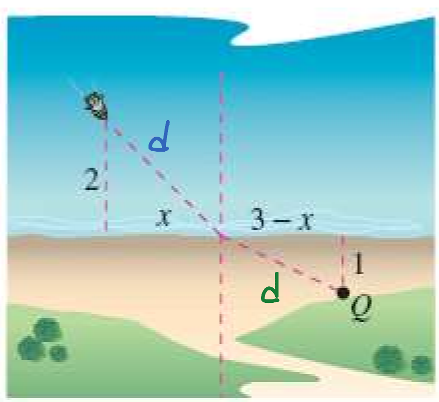
$w = \sqrt{\frac{400}{3}} = \frac{20}{\sqrt{3}} = \frac{20\sqrt{3}}{3}$

The width should be $\frac{20\sqrt{3}}{3} \approx 11.5$ in and height

$h^2 = 400 - \frac{400}{3} = \frac{800}{3}$ or $h = \sqrt{\frac{800}{3}} \approx 16.3$ inches

Strongest implies max strength. $S'' = -6kw$ which is negative for all positive widths and constants k. Concave down $\Rightarrow$ maximum.

Example 8: A man is in a boat 2 miles from the nearest point on the coast. He is to go to a point *Q*, located 3 miles down the coast and 1 mile inland. He can row at 2 miles per hour and walk at 4 miles per hour. Toward what point on the coast should he row in order to reach point *Q* in the least time.



minimize time

dist = rate · time or

$\frac{\text{dist}}{\text{rate}} = \text{time}$

water: $d = \sqrt{x^2 + 4}$

water time $= \frac{\sqrt{x^2+4}}{2}$

land: $d = \sqrt{(3-x)^2 + 1} = \sqrt{x^2 - 6x + 10}$

land time $= \frac{\sqrt{x^2-6x+10}}{4}$

$$\text{Total time} = \frac{\sqrt{x^2+4}}{2} + \frac{\sqrt{x^2-6x+10}}{4} = \frac{1}{2}\sqrt{x^2+4} + \frac{1}{4}\sqrt{x^2-6x+10}$$

$$T' = \frac{1}{2}\left(\frac{1}{2}(x^2+4)^{-1/2}(2x)\right) + \frac{1}{4}\left(\frac{1}{2}(x^2-6x+10)^{-1/2}(2x-6)\right)$$

$$T' = \frac{x}{2\sqrt{x^2+4}} + \frac{x-3}{4\sqrt{x^2-6x+10}}$$

$T' = 0$ gives

$$0 = \frac{x}{2\sqrt{x^2+4}} + \frac{x-3}{4\sqrt{x^2-6x+10}}$$

$$\left(\frac{-x}{2\sqrt{x^2+4}}\right)^2 = \left(\frac{x-3}{4\sqrt{x^2-6x+10}}\right)^2$$

$$4 \cdot \frac{x^2}{\cancel{4}(x^2+4)} = \frac{x^2-6x+9}{\underset{4}{\cancel{16}}(x^2-6x+10)} \cdot 4$$

$$\frac{x^2}{x^2+4} = \frac{x^2-6x+9}{4(x^2-6x+10)}$$ Cross multiply

$$4x^4 - 24x^3 + 40x^2 = x^4 - 6x^3 + 9x^2 + 4x^2 - 24x + 36$$

$$3x^4 - 18x^3 + 27x^2 + 24x - 36 = 0$$ divide by 3

$$x^4 - 6x^3 + 9x^2 + 8x - 12 = 0$$

$x = 1$ or $x \approx -1.11$

not going backwards

The man should row toward a point one mile down the coast in order to minimize his travel time.