3.8 Newton's Method

Newton's Method for Approximating the Zeros of a Function – Let f(c) = 0, where *f* is differentiable on an open interval containing *c*. Then, to approximate *c*, use the following steps.

1. Make an initial estimate x_1 that is close to *c*. (A graph is helpful.)

2. Determine a new approximation $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$.

3. If $|x_n - x_{n+1}|$ is within the desired accuracy, let x_{n+1} serve as the final approximation. Otherwise, return to step 2 and calculate a new approximation.

Each successive application of the procedure is called iteration. Not that this is the same process your graphing calculator uses when asked to find a zero.

Examples: Complete two iterations of Newton's Method for the function using the given initial guess.

1. $f(x) = x^3 - 3,\ x_1 = 1.4$

Solution: First, find the derivative as we need it for the formula. $f'(x) = 3x^2$.

For $x_1 = 1.4$ we will have $x_2 = 1.4 - \dfrac{f(1.4)}{f'(1.4)} = 1.4 - \dfrac{(1.4)^3 - 3}{3(1.4)^2} = 1.4435$

Now $x_2 = 1.4435$ and $x_3 = 1.4435 - \dfrac{f(1.4435)}{f'(1.4435)} = 1.4435 - \dfrac{(1.4435)^3 - 3}{3(1.4435)^2} = 1.4423$

We see that both x_1 and x_2 are accurate to two decimal places so our answer would be 1.44.

2. $f(x) = \tan x,\ x_1 = 0.1$

Solution: The derivative is $f'(x) = \sec^2 x$. Using the formula with $x_1 = 0.1$ (and in radian mode), we get $x_2 = 0.1 - \dfrac{\tan(0.1)}{\sec^2(0.1)} = 6.6533 \times 10^{-4} = 0.00066533$. Next we can find $x_3 = 0.00066533 - \dfrac{\tan(0.00066533)}{\sec^2(0.00066533)} = 1.96 \times 10^{-10} = 0.000000000196$. With this result we are pretty sure that we have found our solution.

Examples: Approximate the zero(s) of the function. Use Newton's Method and continue the process until two successive approximations differ by less than 0.001.

1. $f(x) = x^5 + x - 1$ it looks like there is a zero just below x = 1.

Solution: The derivative is $f'(x) = 5x^4 + 1$.

$$x_1 = 1$$

$$x_2 = 1 - \frac{f(1)}{f'(1)} = \frac{5}{6} \approx 0.83333$$

$$x_3 = \frac{5}{6} - \frac{f(5/6)}{f'(5/6)} = \frac{10138}{13263} \approx 0.76438$$

$$x_4 = 0.76438 - \frac{f(0.76438)}{f'(0.76438)} = 0.75502$$

$x_5 = 0.75502 - \dfrac{f(0.75502)}{f'(0.75502)} = 0.75487770177$, we're getting close…

$$x_6 = 0.75488 - \frac{f(0.75488)}{f'(0.75488)} = 0.754877666247$$

These last two values are within 0.001 of each other so we will accept x_6 as our solution.

2. $f(x) = x - 2\sqrt{x+1}$, it looks like there is a zero near x = 5.

Solution: First we start with the derivative which is $f'(x) = 1 - \frac{1}{\sqrt{x+1}}$.

$$x_1 = 5$$

$$x_2 = 5 - \frac{f(5)}{f'(5)} = 4.8292856399$$

$$x_3 = 4.8292856399 - \frac{f(4.8292856399)}{f'(4.8292856399)} = 4.8284271471$$

$$x_4 = 4.8284271471 - \frac{f(4.8284271471)}{f'(4.8284271471)} = 4.82842712475$$

These last two approximations are similar to seven decimal places. We will take x_4 as our approximation of the zero of the given function.

There is a video quiz for this topic; watching the video will help the process become clearer if you need to see it happen. There is also a PowerPoint located on my website at http://www.math.utep.edu/Faculty/tuesdayj/math1411/newtonsmethodppt.pdf if you need more instruction.