4.2 Area

Sigma Notation – The sum of *n* terms $a_1, a_2, a_3, \dots, a_n$ is written as $\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \dots + a_n$ where *I* is the index of summation, a_i is the *i*th term of the sum, and the upper and lower bounds of summation are *n* and 1.

Examples: Find the sum.

1. $\sum_{i=1}^{6}(3i+2) = (3(1)+2) + (3(2)+2) + (3(3)+2) + (3(4)+2) + (3(5)+2) + (3(6)+2)$
$= 5 + 8 + 11 + 14 + 17 + 20$
$= 13 + 25 + 37$
$= 75$

2. $\sum_{k=5}^{8} k(k-4) = 5(5-4) + 6(6-4) + 7(7-4) + 8(8-4)$
$= 5 + 12 + 21 + 32$
$= 17 + 53$
$= 70$

3. $\sum_{j=4}^{7} \frac{2}{j} = \frac{2}{4} + \frac{2}{5} + \frac{2}{6} + \frac{2}{7} = \frac{319}{210}$

You can see how summation notation makes it easier to write long sums.

Now we're going to take the sum and write in sigma notation
Keep an eye out for what changes - the index of summation and for what doesn't change in order to make the formula

Examples: Use sigma notation to write the sum.

1. $\frac{1}{5(1)}+\frac{1}{5(2)}+\frac{1}{5(3)}+\ldots+\frac{1}{5(11)}$ $= \sum_{i=1}^{11} \frac{1}{5i}$

2. $\left[1-\left(\frac{1}{4}\right)^2\right]+\left[1-\left(\frac{2}{4}\right)^2\right]+\ldots+\left[1-\left(\frac{4}{4}\right)^2\right]$ $= \sum_{i=1}^{4}\left[1-\left(\frac{i}{4}\right)^2\right]$

3. $\left[2\left(1+\frac{3}{n}\right)^2\right]\left(\frac{3}{n}\right)+\ldots+\left[2\left(1+\frac{3n}{n}\right)^2\right]\left(\frac{3}{n}\right)$ $= \sum_{i=3}^{3n}\left[2\left(1+\frac{i}{n}\right)^2\right]\left(\frac{3}{n}\right)$

The 3 and 3n are only differences in the two expressions

Theorem 4.2 Summation Formulas and properties–

1. $\sum_{i=1}^{n} c = cn$

2. $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

3. $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

4. $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

5. $\sum_{i=1}^{n} ka_i = k\sum_{i=1}^{n} a_i$

6. $\sum_{i=1}^{n}(a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$

Examples: Use the properties of summation and theorem 4.2 to evaluate the sum.

1. $\sum_{i=1}^{30} -18$ $= -18(30) = -540$

Property 1

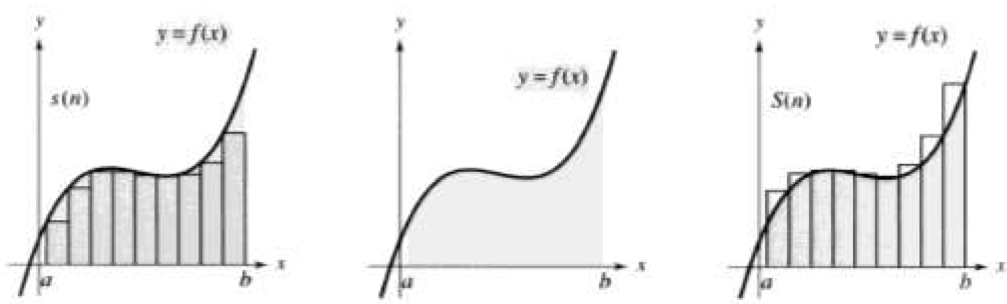
2. $\sum_{i=1}^{16}(5i-4) = \sum_{i=1}^{16} 5i - \sum_{i=1}^{16} 4 = 5\frac{16(17)}{2} - 4(16) = 5(8)(17) - 4(16) = 616$

prop 2+5 prop 1

3. $\sum_{i=1}^{10} i(i^2+1) = \sum_{i=1}^{10}(i^3+i) = \sum_{i=1}^{10} i^3 + \sum_{i=1}^{10} i = \frac{10^2(11)^2}{4} + \frac{10(11)}{2} = 3025 + 55 = 3080$

prop4 prop 2

Why do we care? The antiderivative is a measure of the area of the region between the curve and the x-axis. In order to approximate an area we use easier shapes that we know formulas for. For example, to find the area of a hexagon, we break it into 6 equal triangles because the triangle area formula is easy. Archimedes famously approximated pi by using smaller and smaller triangles to estimate the area of a circle. We will use a similar procedure with rectangles to approximate the area under a curve.



Area of inscribed rectangles is less than area of region.

Area of region

Area of circumscribed rectangles is greater than area of region

If we consider the notation $\int f(x)dx$ we can now clearly see a sum of areas with height f(x) and length dx.

This assignment is split into two parts: 4.2 Part 1 consists of four problems using sigma notation as the examples have shown, whereas 4.2 Part 2 uses this idea of upper and lower sums as approximations on the integral. The homework for Part 2 is pure extra credit. If you would like to earn that credit, I suggest reading examples 3 – 8 of the text, beginning on page 257.