4.4 The Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus (TFToC) – If a function f is continuous on the closed interval [a, b] and F is an antiderivative of f on the interval [a, b], then $\int_a^b f(x)\,dx = F(b) - F(a)$.

Examples: Evaluate.

1. $\int_4^9 5\,dv = 5v\Big|_4^9 = 5(9) - 5(4) = 45 - 20 = 25$

evaluated from 4 to 9

2. $\int_2^5 (-3x+4)\,dx = \left(\frac{-3x^2}{2} + 4x\right)\Big|_2^5 = \left(\frac{-3(5)^2}{2} + 4(5)\right) - \left(\frac{-3(2)^2}{2} + 4(2)\right)$

$= \left(\frac{-75}{2} + 20\right) - (-6 + 8) = \frac{-75}{2} + 18 = \frac{-39}{2}$

3. $\int_{-1}^1 (t^3 - 9t)\,dt = \left(\frac{t^4}{4} - \frac{9t^2}{2}\right)\Big|_{-1}^1 = \left(\frac{(1)^4}{4} - \frac{9(1)^2}{2}\right) - \left(\frac{(-1)^4}{4} - \frac{9(-1)^2}{2}\right)$

$= \left(\frac{1}{4} - \frac{9}{2}\right) - \left(\frac{1}{4} - \frac{9}{2}\right) = 0$

Note: $y = t^3 - 9t$ is an odd function over interval $[-a, a]$ so we expected 0.

4. $\int_{-2}^{-1} \left(u - \frac{1}{u^2}\right) du = \int_{-2}^{-1} (u - u^{-2})\,du = \left(\frac{u^2}{2} - \frac{u^{-1}}{-1}\right)\Big|_{-2}^{-1} = \left(\frac{(-1)^2}{2} + \frac{1}{(-1)}\right) - \left(\frac{(-2)^2}{2} + \frac{1}{(-2)}\right)$

$= \left(\frac{1}{2} - 1\right) - \left(2 - \frac{1}{2}\right) = -2$

5. $\int_0^2 (2-x)\sqrt{x}\,dx = \int_0^2 (2x^{1/2} - x^{3/2})\,dx = \left(2\frac{x^{3/2}}{\frac{3}{2}} - \frac{x^{5/2}}{\frac{5}{2}}\right)\Big|_0^2 = \left(\frac{4}{3}\sqrt{x^3} - \frac{2}{5}\sqrt{x^5}\right)\Big|_0^2$

still no product rule

$(2-x)x^{1/2} = 2x^{1/2} - x^{3/2}$

$= \left(\frac{4}{3}\sqrt{(2)^3} - \frac{2}{5}\sqrt{(2)^5}\right) - 0$

$= \frac{4}{3}\sqrt{8} - \frac{2}{5}\sqrt{32}$

$= \frac{8}{3}\sqrt{2} - \frac{8}{5}\sqrt{2}$

$= \sqrt{2}\left(\frac{8}{3} - \frac{8}{5}\right) = \frac{16\sqrt{2}}{15}$

Examples: Evaluate.

1. $\int_0^{\pi}(2+\cos x)\,dx = (2x+\sin x)\Big|_0^{\pi} = (2\pi+\sin\pi)-(2(0)+\sin 0)$

$= (2\pi+0)-(0+0)$

$= 2\pi$

2. $\int_0^{\pi/4}\frac{\sec^2\theta}{\tan^2\theta+1}\,d\theta = \int_0^{\pi/4}\frac{\sec^2\theta}{\sec^2\theta}\,d\theta = \int_0^{\pi/4}1\,d\theta = \theta\Big|_0^{\pi/4} = \pi/4 - 0 = \frac{\pi}{4}$

No quotient rule but we do know trig identities and $\tan^2\theta+1=\sec^2\theta$

3. $\int_{-\pi/3}^{\pi/3}4\sec\theta\tan\theta\,d\theta = 4\sec\theta\Big|_{-\pi/3}^{\pi/3} = 4\sec\left(\frac{\pi}{3}\right)-4\sec\left(-\frac{\pi}{3}\right) = 4(2)-4(2)=0$

Examples: Find the area of the region bounded by the graphs of the equations.

1. $\underbrace{y=5x^2+2}_{\text{function}},\ \underbrace{x=0,\ x=2}_{\text{limits}},\ \underbrace{y=0}_{\text{above x-axis}}$

$\int_0^2(5x^2+2)\,dx = \left(\frac{5x^3}{3}+2x\right)\Big|_0^2$

$= \left(\frac{5(2)^3}{3}+2(2)\right)-(0+0) = \frac{40}{3}+4 = \frac{52}{3}$ *

exact, not decimal approximation

2. $y=1+\sqrt[3]{x},\ x=0,\ x=8,\ y=0$

$\int_0^8(1+x^{1/3})\,dx = \left(x+\frac{x^{4/3}}{\frac{4}{3}}\right)\Big|_0^8 = \left(x+\frac{3}{4}x^{4/3}\right)\Big|_0^8 = \left(8+\frac{3}{4}\sqrt[3]{(8)^4}\right)-(0)$

$= 8+\frac{3}{4}(2)^4 = 8+12 = 20$

The Mean Value Theorem for Integrals – If *f* is continuous on the closed interval [a, b], then there exists a number *c* in the closed interval [a, b] such that $\int_a^b f(x)\,dx = f(c)(b-a)$.

Definition of Average Value on an Interval – If *f* is integrable on the closed interval [a, b], then the average value of *f* on the interval is $\frac{1}{b-a}\int_a^b f(x)\,dx$.

Note the average value is f(c) from MVT for Integrals

Examples: Find the value(s) of *c* guaranteed by the MVTfI for the function over the given interval.

1. $f(x)=x^3,\ [0,3]$

$f(c)=c^3$

$b-a=3-0=3$

$\int_0^3 x^3\,dx = c^3(3)$

$\int_0^3 x^3\,dx = \frac{x^4}{4}\Big|_0^3 = \frac{(3)^4}{4} - 0 = \frac{81}{4}$

$\frac{81}{4} = 3c^3$

$\frac{27}{4} = c^3$ so $c=\sqrt[3]{\frac{27}{4}} = \frac{3}{\sqrt[3]{4}}$

2. $f(x)=x-2\sqrt{x},\ [0,2]$

$f(c)=c-2\sqrt{c}$

$b-a=2$

$\int_0^2 (x-2x^{1/2})\,dx = \left(\frac{x^2}{2} - 2\frac{x^{3/2}}{\frac{3}{2}}\right)\Big|_0^2 = \left(\frac{x^2}{2} - \frac{4}{3}\sqrt{x^3}\right)\Big|_0^2 = \left(\frac{(2)^2}{2} - \frac{4}{3}\sqrt{(2)^3}\right) - 0 = 2 - \frac{8}{3}\sqrt{2}$

$2-\frac{8}{3}\sqrt{2} = (c-2\sqrt{c})2$

$1-\frac{4}{3}\sqrt{2} = c-2\sqrt{c}$

$0 = c-2\sqrt{c} - 1 + \frac{4\sqrt{2}}{3}$

using the quadratic formula to find $\sqrt{c}$ then squaring the result gives $c = \frac{\left(\sqrt{-6(2\sqrt{2}-3)} \pm 3\right)^2}{9}$

3. $f(x)=\cos x,\ [-\pi/3,\pi/3]$

$f(c)=\cos c$

$b-a = \frac{\pi}{3} - \left(-\frac{\pi}{3}\right) = \frac{2\pi}{3}$

$\int_{-\pi/3}^{\pi/3} \cos x\,dx = \sin x\Big|_{-\pi/3}^{\pi/3} = \sin\left(\frac{\pi}{3}\right) - \sin\left(-\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} - \left(-\frac{\sqrt{3}}{2}\right)$

$= \sqrt{3}$

$\sqrt{3} = \frac{2\pi}{3}\cos c$

$\frac{3\sqrt{3}}{2\pi} = \cos c$ so $c = \cos^{-1}\left(\frac{3\sqrt{3}}{2\pi}\right) \approx 0.5970577$ in radians as the interval is given

$c \approx 0.597$ rad

Examples: Find the average value of the function over the given interval and all values of x in the interval for which the function equals its average value.

1. $f(x)=9-x^2,\ [-3,3]$

avg value $= f(c) = \dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx$

$$f(c)=\frac{1}{3-(-3)}\int_{-3}^{3}(9-x^2)\,dx=\frac{1}{6}\left(9x-\frac{x^3}{3}\right)\Big|_{-3}^{3}=\frac{1}{6}\left(9(3)-\frac{(3)^3}{3}\right)-\frac{1}{6}\left(9(-3)-\frac{(-3)^3}{3}\right)$$

$$=\frac{1}{6}(27-9)-\frac{1}{6}(-27+9)$$

$$=\frac{1}{6}(18)-\frac{1}{6}(-18)=3+3=6$$

average value $f(c)=6$

$6=9-x^2$
$-3=-x^2$

$x^2=3$

$x=\pm\sqrt{3}$

2. $f(x)=x^3,\ [0,1]$

$f(c)=\dfrac{1}{1-0}\displaystyle\int_0^1 x^3\,dx=1\left(\frac{x^4}{4}\right)\Big|_0^1=\frac{1}{4}$ avg value

$x^3=\frac{1}{4}$

$x=\sqrt[3]{\frac{1}{4}}=\dfrac{1}{\sqrt[3]{4}}=\dfrac{\sqrt[3]{2}}{2}$

ok answer

simplified by rationalizing

3. $f(x)=4x^3-3x^2,\ [-1,2]$

$$f(c)=\frac{1}{2-(-1)}\int_{-1}^{2}(4x^3-3x^2)\,dx=\frac{1}{3}\left(x^4-x^3\right)\Big|_{-1}^{2}=\frac{1}{3}\left((2)^4-(2)^3\right)-\frac{1}{3}\left((-1)^4-(-1)^3\right)$$

$$=\frac{1}{3}(16-8)-\frac{1}{3}(1+1)$$

$$=\frac{8}{3}-\frac{2}{3}=\frac{6}{3}=2$$, the avg value

$4x^3-3x^2=2$ algebra? punt!

Graphically $x=1.137$

4. $f(x)=\sin x,\ [0,\pi]$

$$f(c)=\frac{1}{\pi-0}\int_0^{\pi}\sin x\,dx=\frac{1}{\pi}(-\cos x)\Big|_0^{\pi}=\frac{1}{\pi}(-\cos\pi)-\frac{1}{\pi}(-\cos 0)=\frac{1}{\pi}(-(-1))-\frac{1}{\pi}(-(1))$$
$$=\frac{1}{\pi}+\frac{1}{\pi}=\frac{2}{\pi}$$

$f(c)=\frac{2}{\pi}$, avg value

$\sin x=\frac{2}{\pi}$ gives $x=\sin^{-1}\left(\frac{2}{\pi}\right)\approx 0.6901$ rad

The Second Fundamental Theorem of Calculus – If *f* is continuous on an open interval *I* containing *a*, then, for every *x* in the interval $\frac{d}{dx}\left[\int_a^x f(t)\,dt\right]=f(x)$.

Examples: Use the Second Fundamental Theorem of Calculus to find *F'(x)*.

1. $F(x)=\int_1^x \frac{t^2}{t^2+1}\,dt$

$f(t)$

$F'(x)=\frac{x^2}{x^2+1}$

$f(x)$

2. $F(x)=\int_2^{x^2} \frac{1}{t^3}\,dt$

$f(t)=\frac{1}{t^3}$ so $f(x)=\frac{1}{x^3}$

However the upper limit is x^2 with derivative $2x$

So $F'(x)=\frac{1}{(x^2)^3}\cdot 2x=\frac{2}{x^5}$

3. $F(x)=\int_0^{\sin x} 3t^5\,dt$

$f(u)=3u^5$

so $F'(x)=3\sin^5 x\cos x$

$u=\sin x$ has $du=\cos x$