4.5 Integration by Substitution

Antidifferentiation of a composite function – Let g be a function whose range is an interval I, and let f be a function that is continuous on I. If g is differentiable on its domain and F is an antiderivative of f on I, then $\int f(g(x))g'(x)\,dx = F(g(x))+C$. Letting $u=g(x)$ gives $du=g'(x)\,dx$ and $\int f(u)\,du = F(u)+C$.

To use this theorem we need to remember composition of functions, the chain rule, and the concept of 'inside' function and 'outside' function.

Examples: Identify the inside, the outside, and the derivative of the inside in order to integrate.

1. $\int 2x(x^2+1)^4\,dx$ $= \int (x^2+1)^4\, 2x\,dx = \int u^4\,du = \frac{u^5}{5}+C = \frac{(x^2+1)^5}{5}+C$

in $= x^2+1$

der in $= 2x\,dx$

out $= (\ \ )^4$

$u = x^2+1$

$du = 2x\,dx$

2. $\int 3x^2\sqrt{x^3+1}\,dx$ $= \int (x^3+1)^{1/2}\, 3x^2\,dx = \int u^{1/2}\,du = \frac{u^{3/2}}{\frac{3}{2}}+C = \frac{2}{3}u^{3/2}+C$

$= \frac{2}{3}\sqrt{(x^3+1)^3}+C$

in $= x^3+1$

der in $= 3x^2\,dx$

out $= \sqrt{\ \ } = (\ \ )^{1/2}$

$u = x^3+1$

$du = 3x^2\,dx$

3. $\int \sec^2 x(\tan x+3)\,dx$ $= \int (\tan x+3)\sec^2 x\,dx = \int u\,du = \frac{u^2}{2}+C = \frac{(\tan x+3)^2}{2}+C$

in $= \tan x+3$

der in $= \sec^2 x\,dx$

out $= (\ \ )^1$

$u = \tan x+3$

$du = \sec^2 x\,dx$

4. $\int x(x^2-5)^7\,dx = \int (x^2-5)^7\, x\,dx = \frac{1}{2}\int u^7\,du = \frac{1}{2}\frac{u^8}{8} + C = \frac{1}{16}u^8 + C$

$= \frac{1}{16}(x^2-5)^8 + C$

in $= x^2-5$

der in $= 2x\,dx$

out $\overset{?}{=} (\ )^7$

$u = x^2-5$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

5. $\int x^2(2x^3-1)^{3/2}\,dx = \int (2x^3-1)^{3/2}\, x^2\,dx = \frac{1}{6}\int u^{3/2}\,du = \frac{1}{6}\frac{u^{5/2}}{5/2} + C$

$= \frac{1}{6}\cdot\frac{2}{5}u^{5/2} + C$

$= \frac{1}{15}(2x^3-1)^{5/2} + C$

in $= 2x^3-1$

der in $= 6x^2\,dx$

out $= (\ )^{3/2}$

$u = 2x^3-1$

$du = 6x^2\,dx$

$\frac{1}{6}du = x^2\,dx$

6. $\int \frac{x^2}{(16-x^3)^2}\,dx = \int (16-x^3)^{-2}\, x^2\,dx = -\frac{1}{3}\int u^{-2}\,du = -\frac{1}{3}\frac{u^{-1}}{-1} + C$

$= \frac{1}{3}\cdot\frac{1}{u} + C = \frac{1}{3(16-x^3)} + C$

in $= 16-x^3$

der in $= -3x^2\,dx$

out $= \frac{1}{(\ )^2} = (\ )^{-2}$

$u = 16-x^3$

$du = -3x^2\,dx$

$-\frac{1}{3}du = x^2\,dx$

7. $\int \cos 8x\,dx = \int \cos(8x)\,dx = \frac{1}{8}\int \cos u\,du = \frac{1}{8}\sin u + C = \frac{1}{8}\sin(8x) + C$

in $= 8x$

der in $= 8\,dx$

out $= \cos(\ )$

$u = 8x$

$du = 8\,dx$

$\frac{1}{8}du = dx$

8. $\int \frac{\sin x}{\cos^3 x}\,dx = \int (\cos x)^{-3}\,\sin x\,dx = -\int u^{-3}\,du = -\frac{u^{-2}}{-2} + C$

$= \frac{1}{2u^2} + C$

$= \frac{1}{2\cos^2 x} + C$

Inside is tricky, in general we won't "create" derivatives for the denominator so let

in $= \cos x$ der in $= -\sin x\,dx$

out $= \frac{1}{(\ )^3} = (\ )^{-3}$

$u = \cos x$

$du = -\sin x\,dx$

$-du = \sin x\,dx$

9. $\int x\sqrt{x+6}\,dx$

$in = x+6$

$\text{der } in = 1\,dx$

$\text{outside} = \sqrt{\ }$

our regular substitution method will not handle the x outside the radical. When this happens we do a double substitution that acts as a change of variable.

Let $u = x+6$

Then $u-6 = x$

$du = dx$

$$\int x^{*}\sqrt{x+6}\,dx = \int (u-6)^{*}u^{1/2}\,du$$

$$= \int (u^{3/2} - 6u^{1/2})\,du$$

$$= \frac{2}{5}u^{5/2} - 6\frac{2}{3}u^{3/2} + C$$

$$= \frac{2}{5}(x+6)^{5/2} - 4(x+6)^{3/2} + C$$

*we couldn't distribute with a sum under the radical, but the radical does distribute over the difference.

10. $\int (x+1)\sqrt{2-x}\,dx$ similar to #9

$u = 2-x \quad du = -dx$

$x = 2-u$

$$\int (x+1)(2-x)^{1/2}\,dx = -\int (2-u+1)u^{1/2}\,du$$

$$= -\int (3u^{1/2} - u^{3/2})\,du$$

$$= -\left(3\cdot\frac{2}{3}u^{3/2} - \frac{2}{5}u^{5/2}\right) + C$$

$$= \frac{2}{5}(2-x)^{5/2} - 2(2-x)^{3/2} + C$$

Examples: Evaluate the definite integral using substitution.

1. $\int_{-2}^{4} x^2\left(x^3+8\right)^2 dx$

when using substitution with definite integrals, be sure to also change limits of integration

$u = x^3+8$

$du = 3x^2\,dx$

$\frac{1}{3}du = x^2\,dx$

when $x = -2$

$u = (-2)^3 + 8 = 0$

when $x = 4$

$u = (4)^3 + 8 = 72$

$$\frac{1}{3}\int_0^{72} u^2\,du = \frac{1}{3}\frac{u^3}{3}\Big|_0^{72}$$

$$= \frac{1}{9}(72)^3 - \frac{1}{9}(0)^3$$

$$= \frac{1}{9}(72)^3 = \boxed{41472}$$

notice you do not change back to x from u when limits are also changed

2. $\int_0^1 x\sqrt{1-x^2}\,dx$

$= -\frac{1}{2}\int_1^0 u^{1/2}\,du = -\frac{1}{2}\frac{2}{3}u^{3/2}\Big|_1^0$

$u = 1-x^2$

$du = -2x\,dx$

$-\frac{1}{2}du = x\,dx$

when $x=0$
$u=1$
when $x=1$
$u=0$

$= -\frac{1}{3}\sqrt{u^3}\Big|_1^0$

$= \left(-\frac{1}{3}\sqrt{(0)^3}\right) - \left(-\frac{1}{3}\sqrt{(1)^3}\right)$

$= 0 + \frac{1}{3} = \boxed{\frac{1}{3}}$

3. $\int_0^2 \frac{x}{\sqrt{1+2x^2}}\,dx$

$= \frac{1}{4}\int_1^9 u^{-1/2}\,du = \frac{1}{4}\frac{2}{1}u^{1/2}\Big|_1^9$

$u = 1+2x^2$

$du = 4x\,dx$

$\frac{1}{4}du = x\,dx$

$x=0, u=1$

$x=2, u=9$

$= \frac{1}{2}\sqrt{u}\Big|_1^9 = \frac{1}{2}\sqrt{9} - \frac{1}{2}\sqrt{1}$

$= \frac{3}{2} - \frac{1}{2} = \boxed{1}$

Fact: Let *f* be integrable on the closed interval [-a, a].

1. If *f* is an even function, then $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$

2. If *f* is an odd function, then $\int_{-a}^{a} f(x)\,dx = 0$. we saw this in previous examples!