4.6 Numerical Integration

Sometimes we run into integrals that appear easy but just do not have "easy" antiderivatives. In cases such as this, we turn to numerical integration to approximate the definite integral.

The Trapezoidal Rule – Let *f* be continuous on [*a*,*b*]. The Trapezoidal Rule for approximating $\int_a^b f(x)\,dx$ is given by

$$\int_a^b f(x)\,dx \approx \frac{b-a}{2n}\left[f(x_0)+2f(x_1)+2f(x_2)+\ldots+2f(x_{n-1})+f(x_n)\right]$$

Moreover, as $n \to \infty$, the right hand side approaches the definite integral.

Example: Use the Trapezoidal Rule to approximate $\int_0^\pi \sin x\,dx$ using *n = 4* and *n = 8*. Compare this to the actual value.

Solution: When *n = 4*, $\Delta x = \pi/4$, and you obtain

$$\int_0^\pi \sin x\,dx \approx \frac{\pi}{8}\left(\sin 0 + 2\sin\frac{\pi}{4} + 2\sin\frac{\pi}{2} + 2\sin\frac{3\pi}{4} + \sin\pi\right)$$

$$= \frac{\pi}{8}\left(0+\sqrt{2}+2+\sqrt{2}+0\right) = \frac{\pi\left(1+\sqrt{2}\right)}{4} \approx 1.896$$

Similarly, when *n = 8,* $\Delta x = \frac{\pi}{8}$, and you obtain

$$\int_0^\pi \sin x\,dx \approx \frac{\pi}{16}\left(\sin 0 + 2\sin\frac{\pi}{8} + 2\sin\frac{\pi}{4} + \ldots + 2\sin\frac{7\pi}{8} + \sin\pi\right)$$

$$= \frac{\pi}{16}\left(2+2\sqrt{2}+4\sin\frac{\pi}{8}+4\sin\frac{3\pi}{8}\right) \approx 1.974$$

Since this is an integral we can actually find, we compare the actual value to find

$$\int_0^\pi \sin x\,dx = -\cos x^{\pi}{}_{0} = -\cos\pi - \left(-\cos 0\right) = -\left(-1\right)+1 = 2$$

We see that as n became larger, and we had more subintervals for approximation, our solution was closer to the actual value of the integral.

A second method for approximating is called Simpson's Rule and is named after the English mathematician Thomas Simpson.

Simpson's Rule – Let f be continuous on $[a,b]$ and let n be an even integer. Simpson's Rule for approximating $\int_a^b f(x)\,dx$ is

$$\int_a^b f(x)\,dx \approx \frac{b-a}{3n}\left[f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+\ldots+4f(x_{n-1})+f(x_n)\right]$$

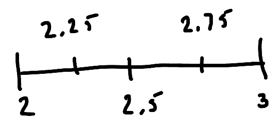
Moreover, as $n \to \infty$, the right hand side approaches the definite integral.

As these are just approximations, it would be nice to know how close we are to being correct. We do have an error analysis available, but if you want to know more about that take a numerical analysis class where that is the focus. Here at UTEP that is Math 4329 and Math 5329.

Example: Use Simpson's Rule and the Trapezoidal Rule to estimate the derivative then compare it to the actual value. Round all answers to four decimal places.

1. $\int_2^3 \frac{2}{x^2}\,dx, \quad n=4$

Partition $[2,3]$ into four intervals



Trapezoidal

$$\int_2^3 \frac{2}{x^2}\,dx = \frac{3-2}{2(4)}\left[f(2)+2f(2.25)+2f(2.5)+2f(2.75)+f(3)\right]$$

$$= \frac{1}{8}\left[\frac{1}{2}+\frac{4}{(2.25)^2}+\frac{4}{(2.5)^2}+\frac{4}{(2.75)^2}+\frac{2}{9}\right] = 0.3352$$

Simpson's

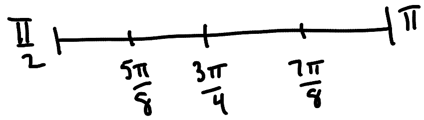
$$\int_2^3 \frac{2}{x^2}\,dx = \frac{3-2}{3(4)}\left[f(2)+4f(2.25)+2f(2.5)+4f(2.75)+f(3)\right]$$

$$= \frac{1}{12}\left[\frac{1}{2}+\frac{8}{(2.25)^2}+\frac{4}{(2.5)^2}+\frac{8}{(2.75)^2}+\frac{2}{9}\right] = 0.3334$$

Actual

$$\int_2^3 2x^{-2}\,dx = 2\frac{x^{-1}}{-1}\Big|_2^3 = -\frac{2}{x}\Big|_2^3 = \frac{-2}{3}-\left(\frac{-2}{2}\right) = \frac{1}{3} = 0.3333$$

2. $\int_{\pi/2}^{\pi} \sqrt{x}\sin x\,dx, \quad n = 4$



Trapezoidal

$$\int_{\pi/2}^{\pi} \sqrt{x}\sin x\,dx = \frac{\pi - \pi/2}{2(4)}\left[f\left(\frac{\pi}{2}\right) + 2f\left(\frac{5\pi}{8}\right) + 2f\left(\frac{3\pi}{4}\right) + 2f\left(\frac{7\pi}{8}\right) + f(\pi)\right]$$

$$= \frac{\pi/2}{8}\left[\sqrt{\frac{\pi}{2}}\sin\frac{\pi}{2} + 2\sqrt{\frac{5\pi}{8}}\sin\frac{5\pi}{8} + 2\sqrt{\frac{3\pi}{4}}\sin\frac{3\pi}{4} + 2\sqrt{\frac{7\pi}{8}}\sin\frac{7\pi}{8} + \sqrt{\pi}\sin\pi\right]$$

$$= \frac{\pi}{16}\left[7.28224823945\right] = 1.4299$$

Simpson's

$$\int_{\pi/2}^{\pi} \sqrt{x}\sin x\,dx = \frac{\pi - \pi/2}{3(4)}\left[f(\pi/2) + 4f\left(\frac{5\pi}{8}\right) + 2f\left(\frac{3\pi}{4}\right) + 4f\left(\frac{7\pi}{8}\right) + f(\pi)\right]$$

$$= \frac{\pi}{24}\left[\sqrt{\pi/2}\sin\frac{\pi}{2} + 4\sqrt{\frac{5\pi}{8}}\sin\frac{5\pi}{8} + 2\sqrt{\frac{3\pi}{4}}\sin\frac{3\pi}{4} + 4\sqrt{\frac{7\pi}{8}}\sin\frac{7\pi}{8} + \sqrt{\pi}\sin\pi\right]$$

$$= 1.4583$$

Actual? This is not possible by hand. You can get a <u>free</u> graphing calculator for your smartphone by downloading 'wabbitemu' You will need the Rom it asks about. This will drain your phone battery with extended use. Youtube can teach you how to find $\int_{\pi/2}^{\pi} \sqrt{x}\sin x\,dx = 1.4579$ if you need help.