Chapter Five: Logarithmic, Exponential, and Other Transcendental Functions

5.1 The Natural Logarithmic Function: Differentiation

Recall from Pre-Calculus: The domain of any logarithmic function is the set of positive integers. Therefore to find the domain of a logarithmic function, set the argument (the stuff inside the log) greater than zero and solve.

Examples: Find the domain.

1. $f(x) = \log_5(3x-7)$

$3x - 7 > 0$

$3x > 7$

$x > \frac{7}{3}$

$\left(\frac{7}{3}, \infty\right)$

2. $g(x) = \log_{13}(x^2 - 4x)$

$x^2 - 4x > 0$

Solve related equation:

$x^2 - 4x = 0$

$x(x-4) = 0$

$x = 0, x = 4$

test points: -1, 2, 5

$+ \quad 0 \quad - \quad 4 \quad +$

$(-1)^2 - 4(-1) = 1 + 4 = 5 = +$

$(2)^2 - 4(2) = 4 - 8 = -$

$(5)^2 - 4(5) = 25 - 20 = +$

Now select regions where $x^2 - 4x > 0$.

$(-\infty, 0) \cup (4, \infty)$

Definition of the Natural Log Function – The natural logarithmic function is defined by $\ln x = \int_1^x \frac{1}{t}\,dt,\ x > 0$. The domain of the natural log function is the set of all positive real numbers.

Properties of the Natural Logarithmic Function:

1. The domain is (0,∞) and the range is (-∞,∞).

2. The function is continuous, increasing, and one-to-one.

3. The graph is concave downward.

Logarithmic Properties – If *a* and *b* are positive numbers and *n* is rational, then the following properties are true:

1. $\ln(1) = 0$

2. $\ln(ab) = \ln a + \ln b$

3. $\ln(a^n) = n \ln a$

4. $\ln\left(\frac{a}{b}\right) = \ln a - \ln b$

Examples: Use the properties of logarithms to approximate the indicated logarithms, given than $\ln 2 \approx 0.6931$ and $\ln 3 \approx 1.0986$.

1. $\ln 6 = \ln(2\cdot 3) = \ln 2 + \ln 3 \approx 0.6931 + 1.0986 = 1.7917$

2. $\ln \frac{2}{3} = \ln 2 - \ln 3 \approx 0.6931 - 1.0986 = -0.4055$

3. $\ln 81 = \ln(3^4) = 4\ln 3 \approx 4(1.0986) = 4.3944$

4. $\ln \sqrt{3} = \ln(3^{1/2}) = \frac{1}{2}\ln 3 \approx \frac{1}{2}(1.0986) = 0.5493$

Definition of *e* – The letter *e* denotes the positive real number such that $\ln e = \int_1^e \frac{1}{t}\,dt = 1$

Note that
$\ln e = 1$

Examples: Expand the logarithmic expression.

1. $\ln \sqrt{x^5} = \ln(x^5)^{1/2} = \ln x^{5/2} = \frac{5}{2}\ln x$

2. $\ln(3e^2) = \ln 3 + \ln e^2 = \ln 3 + 2\ln e = \ln 3 + 2(1) = \ln(3) + 2$

This is not ln 5. The 2 is outside of the logarithm

Examples: Write as a logarithm of a single quantity.

1. $3\ln x + 2\ln y - 4\ln z$
$$= \ln x^3 + \ln y^2 - \ln z^4$$
$$= \ln x^3 y^2 - \ln z^4$$
$$= \ln \frac{x^3 y^2}{z^4}$$

2. $2[\ln x - \ln(x+1) - \ln(x-1)]$
$$= 2\ln x - 2\ln(x+1) - 2\ln(x-1)$$
$$= \ln x^2 - \ln(x+1)^2 - \ln(x-1)^2$$
$$= \ln \frac{x^2}{(x+1)^2} - \ln(x-1)^2$$
$$= \ln \frac{x^2}{(x+1)^2(x-1)^2}$$

Note 1: $\dfrac{\frac{x^2}{(x+1)^2}}{(x-1)^2} = \dfrac{x^2}{(x+1)^2(x-1)^2}$

Note 2: There are many different, but equal, ways of writing this solution.

Theorem – Let u be a differentiable function of x.

1. $\frac{d}{dx}[\ln x] = \frac{1}{x},\ x > 0$

2. $\frac{d}{dx}[\ln u] = \frac{1}{u}\frac{du}{dx} = \frac{u'}{u},\ u > 0$

Examples: Find the derivative of the function.

1. $f(x) = \ln(3x-1)$

$u = 3x-1$

$u' = 3$

$f'(x) = \frac{3}{3x-1}$

2. $h(x) = \ln(2x^2+1)$

$u = 2x^2+1$

$u' = 4x$

$h'(x) = \frac{4x}{2x^2+1}$

3. $y = x^2 \ln x$

product rule

$y' = x^2 \cdot \frac{1}{x} + (\ln x)(2x) = x + 2x\ln x$

4. $y = \ln(\ln x)$

$u = \ln x$

$u' = \frac{1}{x}$

$y' = \frac{\frac{1}{x}}{\ln x} = \frac{1}{x} \cdot \frac{1}{\ln x} = \frac{1}{x\ln x}$

5. $f(x) = \ln\left(\frac{2x}{x+3}\right)$

$u = \frac{2x}{x+3}$

$u' = \frac{(x+3)(2) - 2x(1)}{(x+3)^2} = \frac{2x+6-2x}{(x+3)^2} = \frac{6}{(x+3)^2}$

divide by fraction is multiply by reciprocal

$f'(x) = \frac{\frac{6}{(x+3)^2}}{\frac{2x}{x+3}} = \frac{6}{(x+3)^2} \cdot \frac{x+3}{2x}$

which simplifies to

$f'(x) = \frac{3}{x(x+3)}$

6. $f(x)=\ln\left(x+\sqrt{4+x^2}\right)$

$u = x+\sqrt{4+x^2}$

$u' = 1+\frac{1}{2}(4+x^2)^{-1/2}(2x)$

$u' = 1+\frac{x}{\sqrt{4+x^2}}$

$$f'(x) = \frac{1+\frac{x}{\sqrt{4+x^2}}}{x+\sqrt{4+x^2}} \cdot \frac{\sqrt{4+x^2}}{\sqrt{4+x^2}} = \frac{\sqrt{4+x^2}+x}{x\sqrt{4+x^2}+4+x^2}$$

7. $y=\ln|\csc x|$

$u=|\csc x|$

$du = -\csc x\cot x$

$$y' = \frac{-\csc x\cot x}{|\csc x|} = -\cot x$$

Examples: Find an equation of the tangent line to the graph of *f* at the given point.

$y=mx+b$

1. $f(x)=3x^2-\ln x,\ (1,3)$

$f'(x) = 6x-\frac{1}{x}$

$m = f'(1) = 6(1)-\frac{1}{(1)} = 6-1 = 5$

$y-3 = 5(x-1)$

$y-3 = 5x-5$

$y = 5x-2$

2. $f(x)=x^3\ln x,\ (1,0)$

$f'(x) = x^3\cdot\frac{1}{x} + (\ln x)(3x^2) = x^2+3x^2\ln x$

$m = f'(1) = (1)^2+3(1)^2\ln(1) = 1+3(0) = 1$

$y-0 = 1(x-1)$

$y = x-1$

Examples: Use implicit differentiation to find *dy/dx*.

1. $\ln xy+5x=30$

$\frac{1}{xy}\left(x\frac{dy}{dx}+y(1)\right)+5=0$

$\frac{x}{xy}\frac{dy}{dx}+\frac{y}{xy}+5=0$

$\frac{1}{y}\frac{dy}{dx} = -5-\frac{1}{x}$

$\frac{dy}{dx} = -5y-y/x$

2. $4xy + \ln x^2 y = 7$

$$4x\frac{dy}{dx} + y(4) + \frac{1}{x^2 y}\left(x^2\frac{dy}{dx} + y2x\right) = 0$$

$$4x\frac{dy}{dx} + 4y + \frac{1}{y}\frac{dy}{dx} + \frac{2}{x} = 0$$

$$4x\frac{dy}{dx} + \frac{1}{y}\frac{dy}{dx} = -4y - \frac{2}{x}$$

$$\frac{dy}{dx}\left(4x + \frac{1}{y}\right) = -\left(4y + \frac{2}{x}\right)$$

$$\frac{dy}{dx} = -\frac{4y + \frac{2}{x}}{4x + \frac{1}{y}} \cdot \frac{xy}{xy} = -\frac{4xy^2 + 2y}{4x^2 y + x}$$

Examples: Use logarithmic differentiation to find *dy/dx*.

Logarithmic differentiation is a nice "trick" to handle otherwise difficult derivatives. In this process we take the natural logarithm of both sides of a function equation, use implicit differentiation, then solve for $\frac{dy}{dx}$.

1. $y = x\sqrt{x^2+1},\ x > 0$

product rule with chain rule or...

$\ln y = \ln x\sqrt{x^2+1}$ — take natural log of both sides

$\ln y = \ln x + \frac{1}{2}\ln(x^2+1)$ — use log properties

$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x} + \frac{1}{2}\frac{2x}{x^2+1}$ — find derivatives

Simplify & Solve for dy/dx

$y \cdot \frac{1}{y}\frac{dy}{dx} = \left(\frac{1}{x} + \frac{x}{x^2+1}\right)y$

$\frac{dy}{dx} = \left(\frac{1}{x} + \frac{x}{x^2+1}\right)x\sqrt{x^2+1} = \sqrt{x^2+1} + \frac{x^2\sqrt{x^2+1}}{x^2+1}$

Simplify back to an explicit function of x.

2. $y=\dfrac{x^2\sqrt{3x-2}}{(x+1)^2}$, $x>\dfrac{2}{3}$

$$\ln y=\ln\frac{x^2\sqrt{3x-2}}{(x+1)^2}$$

$$\ln y=\ln x^2+\ln\sqrt{3x-2}-\ln(x+1)^2$$

$$\ln y=2\ln x+\tfrac{1}{2}\ln(3x-2)-2\ln(x+1)$$

$$\frac{1}{y}\frac{dy}{dx}=\frac{2}{x}+\frac{1}{2}\frac{3}{3x-2}-2\frac{1}{x+1}$$

$$\frac{1}{y}\frac{dy}{dx}=\frac{2}{x}+\frac{3}{2(3x-2)}-\frac{2}{x+1}$$

$$\frac{dy}{dx}=\left(\frac{2}{x}+\frac{3}{2(3x-2)}-\frac{2}{x+1}\right)\frac{x^2\sqrt{3x-2}}{(x+1)^2}$$

$$\frac{dy}{dx}=\frac{2x\sqrt{3x-2}}{(x+1)^2}+\frac{3x^2\sqrt{3x-2}}{2(3x-2)(x+1)^2}-\frac{2x^2\sqrt{3x-2}}{(x+1)^3}$$

3. $y=\sqrt{\dfrac{x^2-1}{x^2+1}}$, $x>1$

$$\ln y=\ln\sqrt{\frac{x^2-1}{x^2+1}}$$

$$\ln y=\frac{1}{2}\ln\left(\frac{x^2-1}{x^2+1}\right)$$

$$\ln y=\frac{1}{2}\ln(x^2-1)-\frac{1}{2}\ln(x^2+1)$$

$$\frac{1}{y}\frac{dy}{dx}=\frac{1}{2}\frac{2x}{x^2-1}-\frac{1}{2}\frac{2x}{x^2+1}$$

$$\frac{1}{y}\frac{dy}{dx}=\frac{x}{x^2-1}-\frac{x}{x^2+1}\quad\text{so}\quad\frac{dy}{dx}=\left(\frac{x}{x^2-1}-\frac{x}{x^2+1}\right)\sqrt{\frac{x^2-1}{x^2+1}}$$