5.2 The Natural Logarithmic Function: Integration

Log Rule for Integration – Let u be a differentiable function of x.

1. $\int \frac{1}{x}\,dx = \ln|x| + C$
2. $\int \frac{1}{u}\,du = \ln|u| + C$

* Due to domain restrictions on logarithms, the absolute value is essential to this integral.

Examples: Find the indefinite integral.

1. $\int \frac{10}{x}\,dx = 10\int \frac{1}{x}\,dx = 10\ln|x| + C$

2. $\int \frac{1}{x-5}\,dx = \int \frac{1}{u}\,du = \ln|u| + C = \ln|x-5| + C$

Let $u = x-5$
$du = dx$

3. $\int \frac{1}{4-3x}\,dx = -\frac{1}{3}\int \frac{1}{u}\,du = -\frac{1}{3}\ln|u| + C = -\frac{1}{3}\ln|4-3x| + C$

Let $u = 4-3x$
$du = -3\,dx$
$-\frac{1}{3}\,du = dx$

4. $\int \frac{x^2}{5-x^3}\,dx = -\frac{1}{3}\int \frac{1}{u}\,du = -\frac{1}{3}\ln|u| + C = -\frac{1}{3}\ln|5-x^3| + C$

Let $u = 5-x^3$
$du = -3x^2\,dx$
$-\frac{1}{3}\,du = x^2\,dx$

5. $\int \frac{x^2-2x}{x^3-3x^2}\,dx = \int \frac{1}{x^3-3x^2}(x^2-2x)\,dx = \frac{1}{3}\int \frac{1}{u}\,du = \frac{1}{3}\ln|u|+C$

$= \frac{1}{3}\ln|x^3-3x^2|+C$

Let $u = x^3-3x^2$

$du = (3x^2-6x)\,dx$

$du = 3(x^2-2x)\,dx$

$\frac{1}{3}du = (x^2-2x)\,dx$

6. $\int \frac{1}{x\ln x^3}\,dx = \int \frac{1}{x(3\ln x)}\,dx = \frac{1}{3}\int \frac{1}{x\ln x}\,dx$ Rewrite to determine u.

we let $u = \ln x$

because $du = \frac{1}{x}dx$

$= \frac{1}{3}\int \frac{1}{\ln x}\cdot\frac{1}{x}\,dx = \frac{1}{3}\int \frac{1}{u}\,du$

$= \frac{1}{3}\ln|u|+C$

$= \frac{1}{3}\ln|\ln x|+C$

Don't let all these $\frac{1}{3}$'s fool you, they are coincidence, not a rule.

7. $\int \frac{x^3-4x^2+7x-9}{x-3}\,dx$

First we divide:

	x^3			
3	1	−4	7	−9
		3	−3	12
	1	−1	4	3

$\frac{x^3-4x^2+7x-9}{x-3} = x^2-x+4+\frac{3}{x-3}$

$\int \frac{x^3-4x^2+7x-9}{x-3}\,dx = \int\left(x^2-x+4+\frac{3}{x-3}\right)dx = \frac{x^3}{3}-\frac{x^2}{2}+4x+3\ln|x-3|+C$

Division was necessary as the numerator had a larger degree than the denominator meaning: (1) no way to make u = denom and have u' = numerator and (2) no way to create a du in denominator if u = numerator. Always simplify improper rational expressions to integrate.

Guidelines for Integration:

1. Learn a basic list of integration formulas. Now you know about 12, soon you will know 20.

2. Find an integration formula that resembles all or part of the integrand, and, by trial and error, find a choice of *u* that will make the integrand conform to the formula.

3. If you cannot find a *u*-substitution that works, try altering the integrand. You might try a trig identity, multiplication and division by the same quantity, addition and subtraction of the same quantity, or long division. Be creative.

4. If you have access to computer software that will find anti-derivatives symbolically, use it.

Integrals of the Six Basic Trig Functions –

$$\int \sin u\,du = -\cos u + C \qquad \int \cos u\,du = \sin u + C$$
$$\int \tan u\,du = -\ln|\cos u| + C \qquad \int \cot u\,du = \ln|\sin u| + C$$
$$\int \sec u\,du = \ln|\sec u + \tan u| + C \qquad \int \csc u\,du = -\ln|\csc u + \cot u| + C$$

We already know the top row. Let's take a look at $\int \tan u\,du$ to see if we can find the integral or if it must be memorized. Rewriting we have $\int \tan u\,du = \int \frac{\sin u}{\cos u}du$. Let $w = \cos u$, then $dw = -\sin u\,du$. Now,
$\int \frac{\sin u}{\cos u}du = -\int \frac{1}{w}dw = -\ln|w| + C = -\ln|\cos u| + C$. The good news is that we can find it if we have to. It is much easier to just memorize though.

Examples: Find the indefinite integral.

1. $\int \tan 5\theta\, d\theta$ $= \frac{1}{5}\int \tan u\,du = -\frac{1}{5}\ln|\cos u| + C$

$= -\frac{1}{5}\ln|\cos 5\theta| + C$

Let $u = 5\theta$

$du = 5d\theta$

$\frac{1}{5}du = d\theta$

2. $\int \sec\frac{x}{2}dx$ $= 2\int \sec u\,du = 2\ln|\sec u + \tan u| + C$

$= 2\ln\left|\sec\frac{x}{2} + \tan\frac{x}{2}\right| + C$

Let $u = \frac{x}{2}$

$du = \frac{1}{2}dx$

$2du = dx$

3. $\int \frac{\csc^2 t}{\cot t}dt$ $= \int \frac{1}{\cot t}\cdot \csc^2 t\,dt = -\int \frac{1}{u}du = -\ln|u| + C$

$= -\ln|\cot t| + C$

Let $u = \cot t$

$du = -\csc^2 t\,dt$

$-du = \csc^2 t\,dt$

Examples: Solve the differential equations. Find the equation that passes through the given point.

1. $\frac{dy}{dx} = \frac{x-2}{x}$, $(-1,0)$

Rewrite: $\frac{x-2}{x} = \frac{x}{x} - \frac{2}{x} = 1 - \frac{2}{x}$

$y = \int dy = \int \left(1 - \frac{2}{x}\right)dx$

$y = x - 2\ln|x| + C$

$0 = (-1) - 2\ln|-1| + C$

$0 = -1 - 2(0) + C$

$0 = -1 + C$

$1 = C$

$\boxed{y = x - 2\ln|x| + 1}$

2. $\frac{dr}{dt} = \frac{\sec^2 t}{\tan t + 1}$, $(\pi, 4)$

$r = \int dr = \int \frac{\sec^2 t}{\tan t + 1}dt = \int \frac{1}{u}du = \ln|u| + C$

let $u = \tan t + 1$

$du = \sec^2 t\,dt$

$r = \ln|\tan t + 1| + C$

$4 = \ln|\tan\pi + 1| + C$

$4 = \ln|1| + C$

$4 = C$

$r = \ln|\tan t + 1| + 4$

Examples: Evaluate the definite integral.

1. $\int_{-1}^{1} \frac{1}{2x+3}dx = \frac{1}{2}\int_{1}^{5} \frac{1}{u}\,du = \frac{1}{2}\ln|u|\Big|_{1}^{5} = \frac{1}{2}\ln|5| - \frac{1}{2}\ln|1|$

$= \boxed{\frac{1}{2}\ln|5|}$

let $u = 2x+3$

$du = 2dx$

$\frac{1}{2}du = dx$

$x=-1,\ u=1$

$x=1,\ u=5$

2. $\int_{e}^{e^2} \frac{1}{x\ln x}dx = \int_{1}^{2} \frac{1}{u}\,du = \ln|u|\Big]_{1}^{2} = \ln|2| - \ln|1|$

$= \ln|2|$

let $u = \ln x$

$du = \frac{1}{x}dx$

$x=e,\ u=1$

$x=e^2,\ u=2$

3. $\int_{1}^{2} \frac{1-\cos\theta}{\theta-\sin\theta}d\theta = \int_{1-\sin 1}^{2-\sin 2} \frac{1}{u}\,du = \ln|u|\Big]_{1-\sin 1}^{2-\sin 2} = \ln|2-\sin 2| - \ln|1-\sin 1|$

Let $u = \theta - \sin\theta$

$du = 1-\cos\theta$

$\theta = 1,\ u = 1-\sin 1$

$\theta = 2,\ u = 2-\sin 2$