5.4 Exponential Functions: Differentiation and Integration

Definition of the Natural Exponential Function – The inverse function of the natural logarithmic function $f(x) = \ln x$ is called the natural exponential function and is denoted by $f^{-1}(x) = e^x$. That is, $y = e^x$ if and only if $x = \ln y$.

Note: This is an L, not an I. The word logarithm starts with an L so it should be obvious... Sometimes I find that is not so for everyone.

Properties of the Natural Exponential Function:

1. The domain of $f(x) = e^x$ is $(-\infty, \infty)$, and the range is $(0, \infty)$.

2. The function $f(x) = e^x$ is continuous, increasing, and one-to-one on its entire domain.

3. The graph of $f(x) = e^x$ is concave upward on its entire domain.

4. $\lim_{x\to-\infty} e^x = 0$ and $\lim_{x\to\infty} e^x = \infty$

Operations with Exponential Functions – Let *a* and *b* be any real numbers.

1. $e^a e^b = e^{a+b}$

2. $\dfrac{e^a}{e^b} = e^{a-b}$

Examples: Solve for *x* accurate to three decimal places.

1. $e^{\ln 2x} = 12$

$2x = 12$

$x = 6$

2. $-6 + 3e^x = 8$

$3e^x = 14$

$e^x = \frac{14}{3}$

$x = \ln(14/3)$

$x \approx 1.540$

3. $\ln 4x = 1$

$e^1 = 4x$

$\frac{e^1}{4} = x$

$x \approx 0.67957$

$x \approx 0.680$

Derivatives of Natural Exponential Functions – Let *u* be a differentiable function of *x*.

1. $\dfrac{d}{dx}\left[e^x\right] = e^x$

This is why mathematicians "naturally" use e.

2. $\dfrac{d}{dx}\left[e^u\right] = e^u \dfrac{du}{dx} = u'e^u$

chain rule

derivative of exponent

Examples: Find the derivative.

1. $y = e^{-5x}$

$y' = -5e^{-5x}$

2. $y = e^{-x^2}$

$y' = -2x e^{-x^2}$

3. $y = xe^x$

$y' = xe^x + e^x(1)$

$y' = xe^x + e^x$ this is good

or $y' = e^x(x+1)$ but so is this!

4. $y = x^x e^{-x}$

x^x?!? log differentiation

$\ln y = \ln(x^x e^{-x})$

$\ln y = \ln(x^x) + \ln(e^{-x})$

$\ln y = x\ln x - x\ln e$

$\ln y = x\ln x - x$

$\frac{1}{y}\frac{dy}{dx} = x \cdot \frac{1}{x} + \ln x - 1$

$\frac{dy}{dx} = (1 + \ln x - 1)y$

$\frac{dy}{dx} = (\ln x)x^x e^{-x}$

5. $y = \dfrac{e^x - e^{-x}}{2}$

$y = \frac{1}{2}(e^x - e^{-x})$

$y' = \frac{1}{2}(e^x - (-1e^{-x}))$

$y' = \frac{1}{2}(e^x + e^{-x})$

$y' = \dfrac{e^x + e^{-x}}{2}$

6. $y = \ln e^x$

$y = x\ln e$ (as $\ln e = 1$)

$y = x$

$y' = 1$

Examples: Find the equation of the tangent line to the graph of the function at the given point.

$y - y_1 = m(x - x_1)$ where $m = f'(x_1)$

1. $y = e^{-2x+x^2}$, $(2, 1)$

$y' = (-2+2x)e^{-2x+x^2}$

$m = y'(2) = (-2+2(2))e^{-2(2)+(2)^2}$

$= 2e^0 = 2$

$y - 1 = 2(x - 2)$

$y - 1 = 2x - 4$

$y = 2x - 3$

2. $f(x)=e^3 \ln x,\ (1,0)$

$f'(x)=e^3 \cdot \frac{1}{x}=\frac{e^3}{x}$

$m=f'(1)=\frac{e^3}{1}=e^3$

$y-0=e^3(x-1)$

$y=e^3x-e^3$

Note: e^3 is just a constant

Example: Use implicit differentiation to find *dy/dx* given $e^{xy}+x^2-y^2=10$.

product rule in exponent

$\left(x\frac{dy}{dx}+y(1)\right)e^{xy}+2x-2y\frac{dy}{dx}=0$

$xe^{xy}\frac{dy}{dx}+ye^{xy}+2x-2y\frac{dy}{dx}=0$

$xe^{xy}\frac{dy}{dx}-2y\frac{dy}{dx}=-2x-ye^{xy}$

$\left(xe^{xy}-2y\right)\frac{dy}{dx}=-2x-ye^{xy}$

$\frac{dy}{dx}=\frac{-2x-ye^{xy}}{xe^{xy}-2y}=-\frac{2x+ye^{xy}}{xe^{xy}-2y}$

Example: Find the second derivative of $g(x)=\sqrt{x}+e^x \ln x$

$g'(x)=\frac{1}{2}x^{-1/2}+e^x\cdot\frac{1}{x}+(\ln x)(e^x)=\frac{1}{2}x^{-1/2}+\frac{e^x}{x}+e^x\ln x$

$g''(x)=-\frac{1}{4}x^{-3/2}+\frac{xe^x-e^x}{x^2}+e^x\frac{1}{x}+(\ln x)(e^x)$

$g''(x)=\frac{-1}{4\sqrt{x^3}}+\frac{e^x}{x}-\frac{e^x}{x^2}+\frac{e^x}{x}+e^x\ln x=\frac{-1}{4\sqrt{x^3}}+\frac{2e^x}{x}-\frac{e^x}{x^2}+e^x\ln x$

Notice the copies of the original that appear in the derivatives involving e^x

Integration Rules for Exponential Functions – Let *u* be a differentiable function of *x*.

1. $\int e^x dx = e^x + C$

2. $\int e^u du = e^u + C$

Let it be easy! Do not over-think this.

Examples: Find the indefinite integral.

1. $\int e^{-x^4}\left(-4x^3\right)dx = \int e^u du = e^u + C = e^{-x^4} + C$

Let $u = -x^4$

$du = -4x^3 dx$

natural substitution u = exponent (when possible)

2. $\int e^x\left(e^x+1\right)^2 dx = \int u^2 du = \frac{u^3}{3} + C = \frac{(e^x+1)^3}{3} + C$

u = "inside"

Let $u = e^x + 1$

$du = e^x dx$

3. $\int \frac{e^{2x}+2e^x+1}{e^x}dx = \int (e^x + 2 + e^{-x})dx = e^x + 2x - e^{-x} + C$

Simplify first.

$\frac{e^{2x}}{e^x} + \frac{2e^x}{e^x} + \frac{1}{e^x} = e^x + 2 + e^{-x}$

$u = -x$

$du = -dx$

$-du = dx$

$\int e^{-x}dx = -\int e^u du$

Examples: Evaluate the definite integral.

1. $\int_3^4 e^{3-x}dx = -\int_0^{-1} e^u du = -e^u\Big|_0^{-1} = -e^{-1} - (-e^0)$

$= -\frac{1}{e} + 1$

$= 1 - \frac{1}{e}$

$u = 3 - x$

$du = -dx$

$-du = dx$

$x = 3 \quad u = 0$

$x = 4 \quad u = -1$

2. $\int_0^1 \frac{e^x}{5-e^x}dx = -\int_4^{5-e} \frac{1}{u}du = -\ln|u|\Big|_4^{5-e} = -\ln|5-e| - (-\ln|4|)$

$= \ln 4 - \ln|5-e|$

Let $u = 5 - e^x$

$du = -e^x dx$

$-du = e^x dx$

$x = 0, u = 5 - e^0 = 5 - 1 = 4$

$x = 1, u = 5 - e^1$