5.7 Inverse Trigonometric Functions: Integration

Integrals involving inverse trig functions – Let u be a differentiable function of x, and let $a > 0$.

1. $\int \frac{du}{\sqrt{a^2-u^2}} = \arcsin\frac{u}{a} + C$

2. $\int \frac{du}{a^2+u^2} = \frac{1}{a}\arctan\frac{u}{a} + C$

3. $\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\,arc\sec\frac{|u|}{a} + C$

Why are there only three integrals and not six?

The derivatives of the co-functions were just negatives of these three. This makes the integrals only necessary for the three above. A negative one constant can easily be handled

Examples: Find the integral.

1. $\int \frac{dx}{\sqrt{1-4x^2}}$

radical so must be arcsine or arcsecant, No variable on the outside so must be arcsine. Now we match the form.

$a^2 = 1$ so $a = 1$

$u^2 = 4x^2$ so $u = 2x$

$du = 2dx$

$\frac{1}{2}du = dx$

$$\int \frac{dx}{\sqrt{1-4x^2}} = \frac{1}{2}\int \frac{du}{\sqrt{1-u^2}} = \frac{1}{2}\arcsin\frac{u}{1} + C$$

$$= \frac{1}{2}\arcsin(2x) + C$$

2. $\int \frac{12}{1+9x^2}dx$

no radical ⇒ arctangent

$a^2 = 1$

$u^2 = 9x^2$ so $u = 3x$

$du = 3dx$

$\frac{1}{3}du = dx$

$$12\int \frac{1}{1-9x^2}dx = \frac{12}{3}\int \frac{du}{1+u^2} = 4\cdot\frac{1}{1}\arctan\left(\frac{u}{1}\right) + C$$

$$= 4\arctan(3x) + C$$

3. $\int \frac{1}{4+(x-3)^2}\,dx = \int \frac{1}{4+u^2}\,du = \frac{1}{2}\arctan\left(\frac{u}{2}\right)+C$

$a^2=4,\ a=2$

$= \frac{1}{2}\arctan\left(\frac{x-3}{2}\right)+C$

$u=x-3$

$du=dx$

4. $\int \frac{1}{x\sqrt{x^4-4}}\,dx$

The variable outside the radical indicates arcseacant

$a^2=4,\ a=2$

$u^2=x^4$ so $u=x^2$

$du=2x\,dx$

$\frac{1}{2x}du=dx$

$\int \frac{1}{x\sqrt{x^4-4}}\,dx = \int \frac{1}{2x}\frac{1}{x\sqrt{u^2-4}}\,du$

$= \frac{1}{2}\int \frac{1}{x^2\sqrt{u^2-4}}\,du$

$= \frac{1}{2}\int \frac{1}{u\sqrt{u^2-4}}\,du$

$\frac{1}{4}\text{arcsec}\frac{|x^2|}{2}+C \longleftarrow = \frac{1}{2}\left(\frac{1}{2}\text{arcsec}\frac{|u|}{2}\right)+C$

5. $\int \frac{\sin x}{7+\cos^2 x}\,dx \to \int \frac{-du}{7+u^2} = -\frac{1}{\sqrt{7}}\arctan\left(\frac{u}{\sqrt{7}}\right)+C$

$a^2=7,\ a=\sqrt{7}$

$u=\cos x$

$du=-\sin x\,dx$

$-du=\sin x\,dx$

$= -\frac{1}{\sqrt{7}}\arctan\left(\frac{\cos x}{\sqrt{7}}\right)+C$

6. $\int \frac{4x+3}{\sqrt{1-x^2}}\,dx = \int \frac{4x}{\sqrt{1-x^2}}\,dx + \int \frac{3}{\sqrt{1-x^2}}\,dx$

$$= \boxed{\frac{1}{(1-x^2)^{3/2}} + 3\arcsin x + C}$$

*This is a valuable "trick" in integration. If the whole is too hard, split it into easier parts.

$$\int \frac{4x}{\sqrt{1-x^2}}\,dx = 4\int \frac{-\frac{1}{2}du}{u^{1/2}} = -2\int u^{-1/2}\,du = -2\left(-\tfrac{1}{2}u^{-3/2}\right) = \frac{1}{\sqrt{(1-x^2)^3}}$$

Let $u = 1-x^2$

$du = -2x\,dx$

$-\frac{1}{2}du = x\,dx$

$$\int \frac{3}{\sqrt{1-x^2}}\,dx = 3\arcsin x$$

7. $\int \frac{x-2}{(x+1)^2+4}\,dx$

Separating the subtraction will not help here, please try it to see why. Instead we do a double substitution.

Let $u = x+1$

$du = dx$

and $x = u-1$

$$\int \frac{u-1-2}{u^2+4}\,du = \int \frac{u-3}{u^2+4}\,du = \underset{①}{\int \frac{u}{u^2+4}\,du} - 3\underset{②}{\int \frac{du}{u^2+4}}$$

$$= \frac{1}{2}\ln|u^2+4| - \frac{3}{2}\arctan\frac{u}{2} + C$$

$$= \boxed{\frac{1}{2}\ln|(x+1)^2+4| - \frac{3}{2}\arctan\frac{x+1}{2} + C}$$

① $\int \frac{u}{u^2+4}\,du = \frac{1}{2}\int \frac{1}{w}\,dw = \frac{1}{2}\ln|w| + C$

let $w = u^2+4$

$dw = 2u\,du$

$\frac{1}{2}dw = u\,du$

② $\int \frac{du}{u^2+4} = \frac{1}{2}\arctan\frac{u}{2} + C$

Examples: Evaluate the integral.

1. $\int_0^1 \frac{dx}{\sqrt{4-x^2}} = \arcsin\frac{x}{2}\Big|_0^1 = \arcsin\frac{1}{2} - \arcsin\frac{0}{2}$

$$= \frac{\pi}{6} - 0$$

$$= \frac{\pi}{6}$$

2. $\int_{\sqrt{3}}^{3} \frac{6}{9+x^2}\,dx = 6 \cdot \frac{1}{3} \arctan \frac{x}{3} \Big|_{\sqrt{3}}^{3}$

$= 2\arctan\frac{3}{3} - 2\arctan\frac{\sqrt{3}}{3}$

$= 2\left(\frac{\pi}{4}\right) - 2\left(\frac{\pi}{6}\right)$

$= \frac{\pi}{6}$

3. $\int_{1}^{4} \frac{1}{x\sqrt{16x^2-5}}\,dx = \frac{1}{4}\int_{4}^{16} \frac{1}{\frac{u}{4}\sqrt{u^2-5}}\,du = \frac{1}{4}\int_{4}^{16} \frac{4}{u\sqrt{u^2-5}}\,du$

$u^2 = 16x^2$

so $u = 4x \rightarrow x = \frac{u}{4}$

$du = 4dx$

$\frac{1}{4}du = dx$

$a^2 = 5$ so

$a = \sqrt{5}$

$x=1,\ u=4$

$x=4,\ u=16$

$= \frac{1}{\sqrt{5}} \operatorname{arcsec} \frac{|u|}{\sqrt{5}} \Big|_{4}^{16}$

$= \frac{1}{\sqrt{5}} \operatorname{arcsec} \frac{|16|}{\sqrt{5}} - \frac{1}{\sqrt{5}} \operatorname{arcsec} \frac{|4|}{\sqrt{5}}$