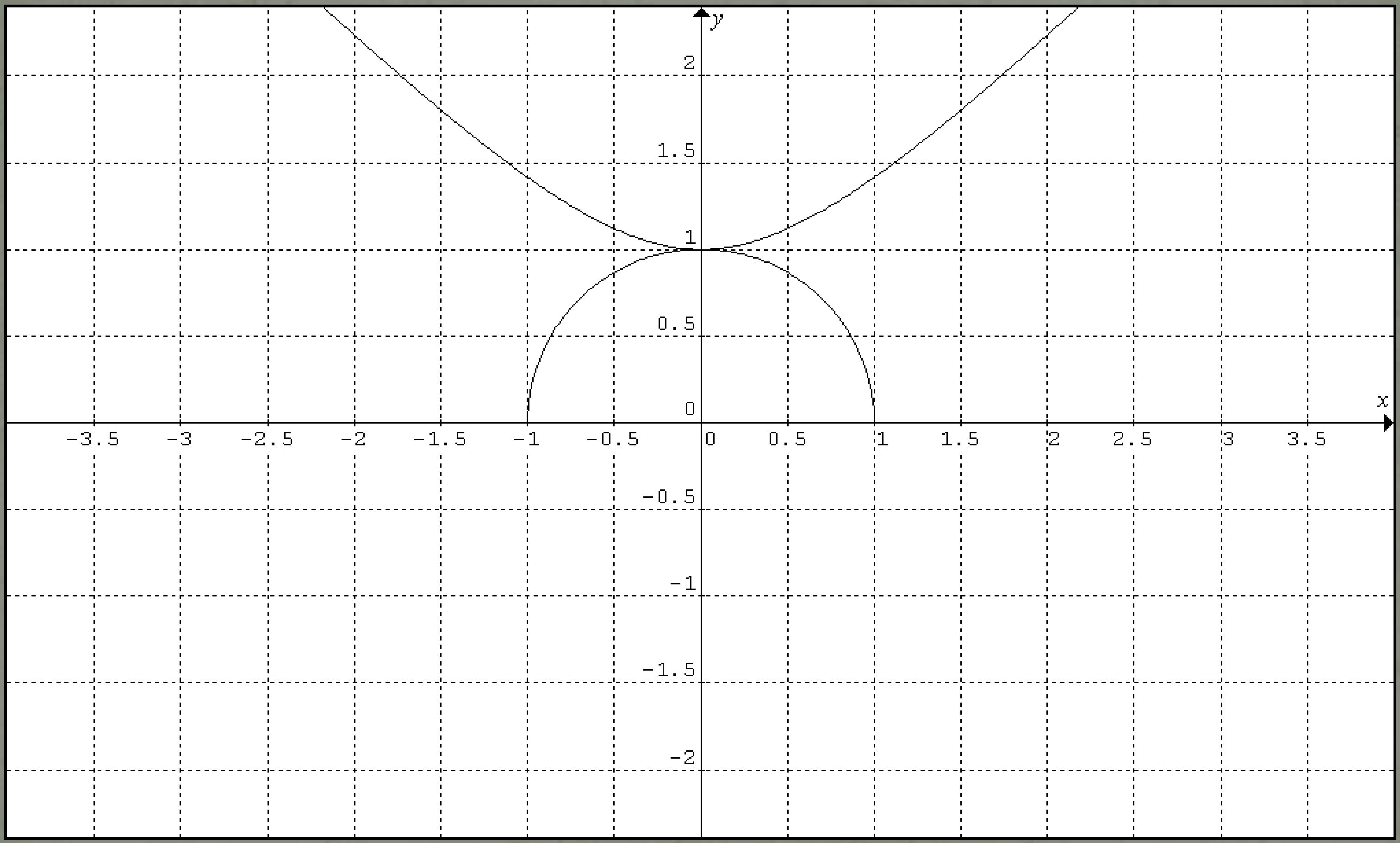
Section 5.8
Hyperbolic Functions:

Hyperbolic function arose from comparison of the area of a semicircular regions with the area of a region under a hyperbola.

Graph of the hyperbola $f(x) = \sqrt{(1+x^2)}$ *and the circle* $g(x) = \sqrt{(1 - x^2)}$



Definitions of the hyperbolic functions

- $$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \operatorname{csch} x = \frac{1}{\sinh x}, x \neq 0$$

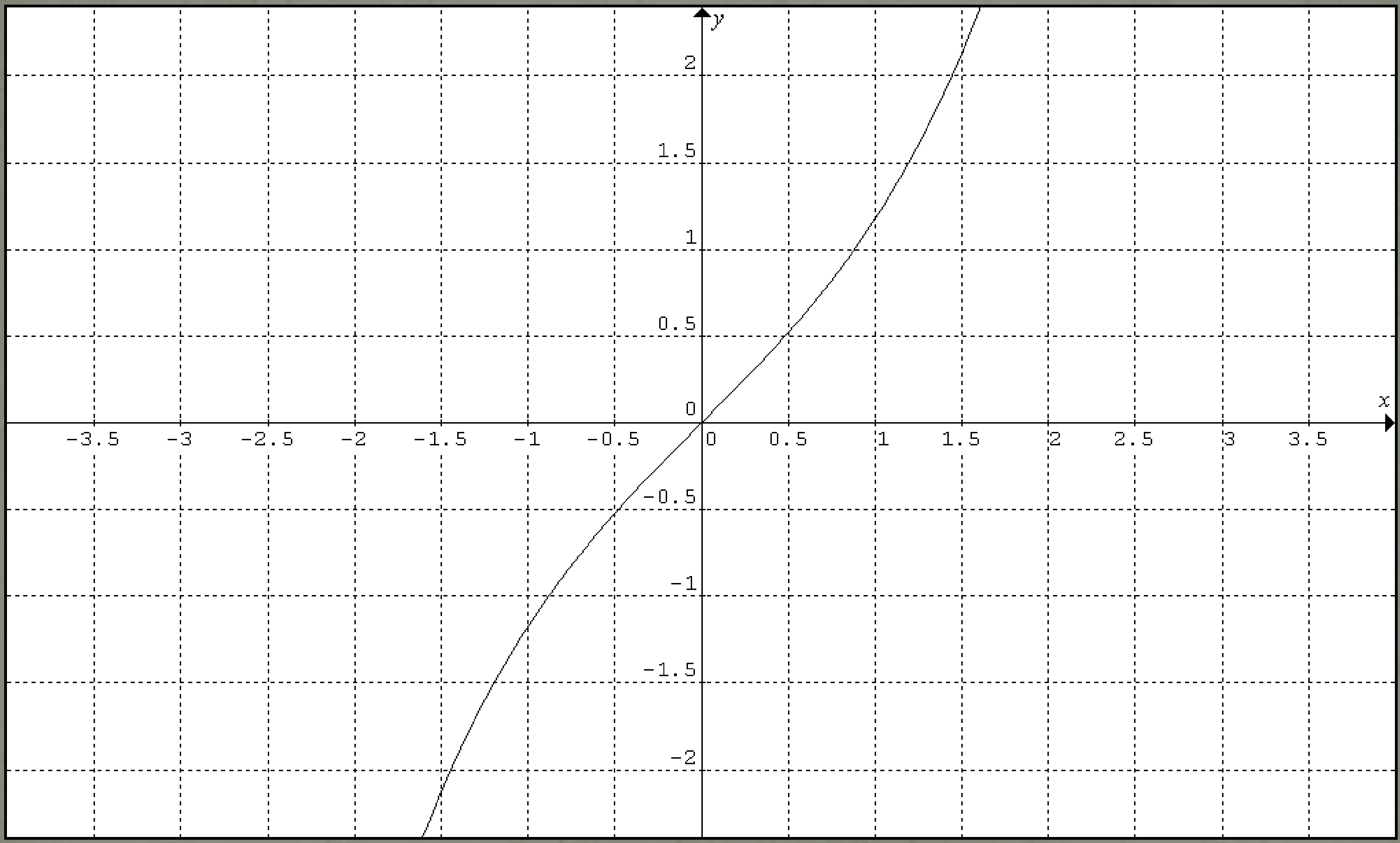
$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \operatorname{sech} x = \frac{1}{\cosh x}$$

$$\tanh x = \frac{\sinh x}{\cosh x} \qquad \coth x = \frac{1}{\tanh x}, x \neq 0$$

Graph of $f(x) = \sinh(x)$

Domain: $(-\infty, \infty)$

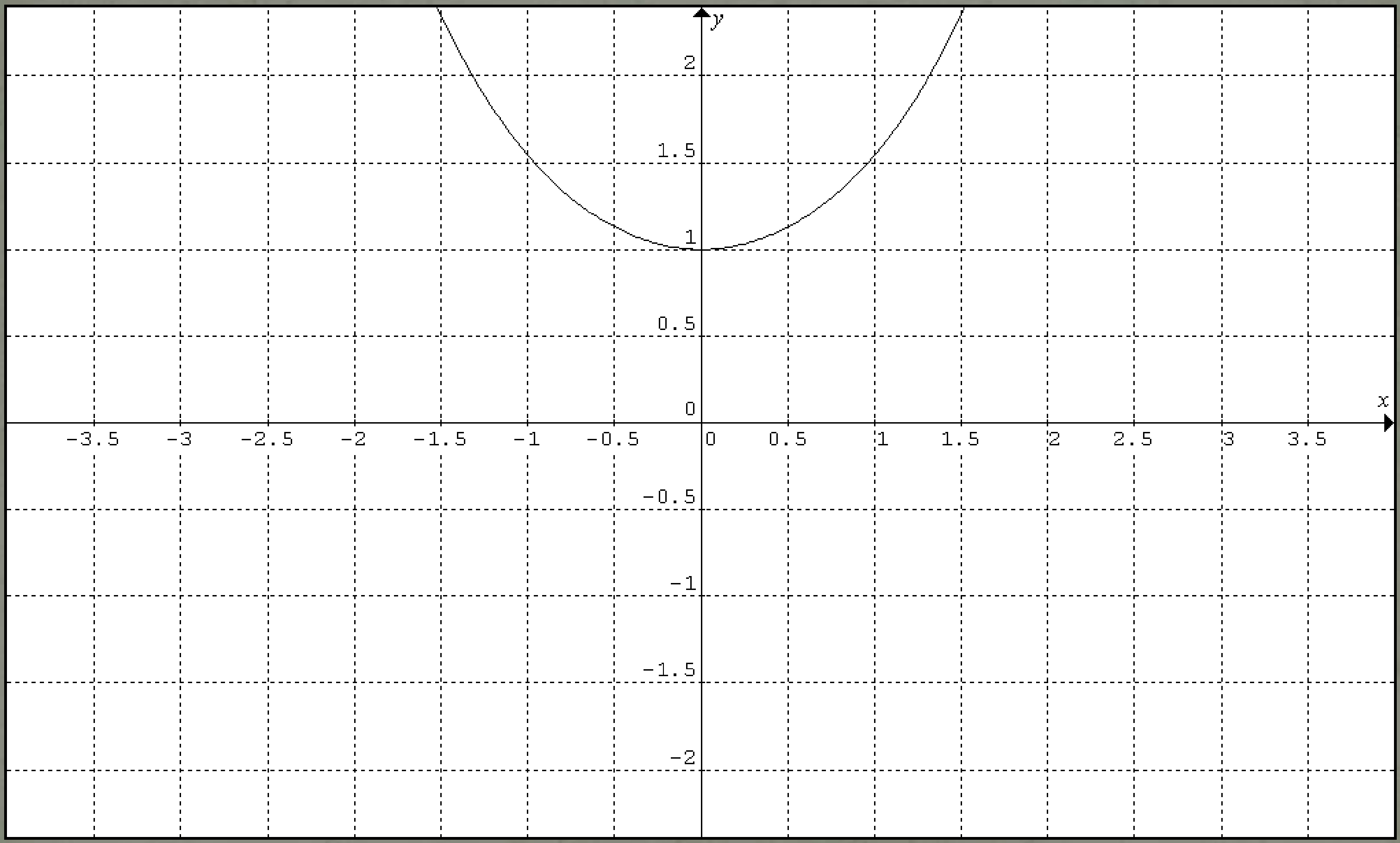
Range: $(-\infty, \infty)$



Graph of $f(x) = \cosh(x)$

Domain: $(-\infty, \infty)$

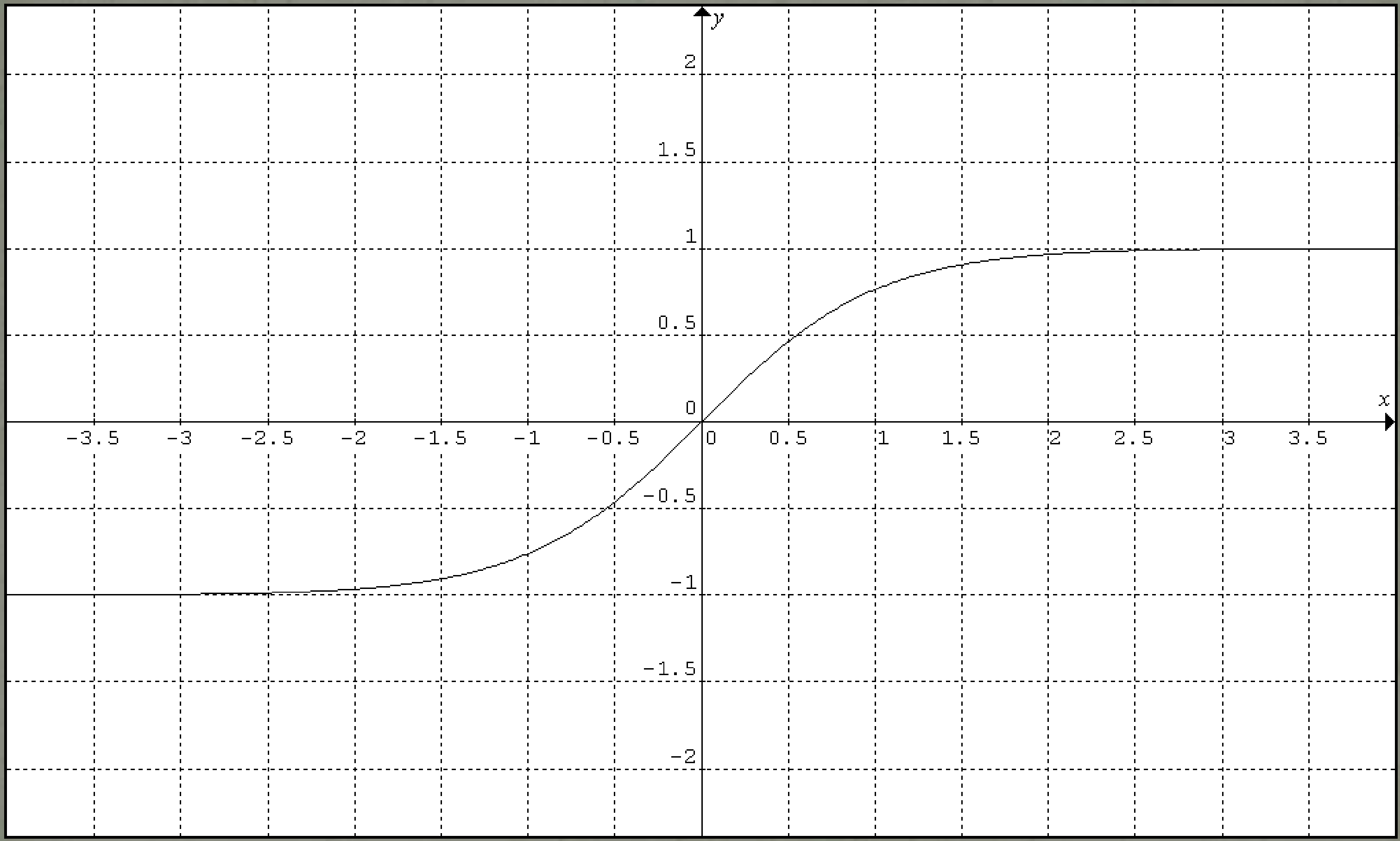
Range: $[1, \infty)$



Graph of $f(x) = tanh\ (x)$

Domain: (-∞, ∞)

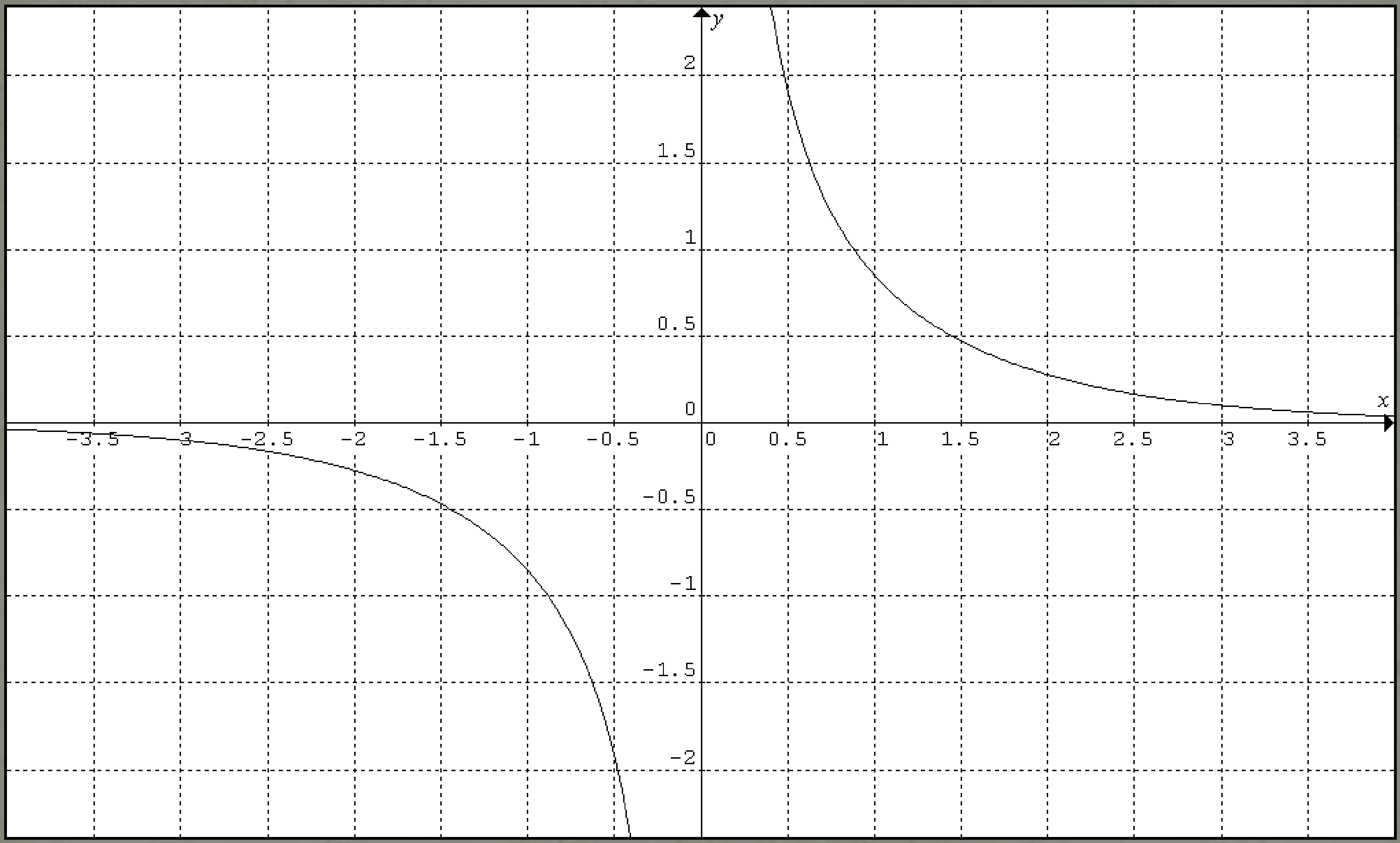
Range: (-1, 1)



Graph of $f(x) = csch\ (x)$

Domain: (-∞, 0)∪(0,∞)

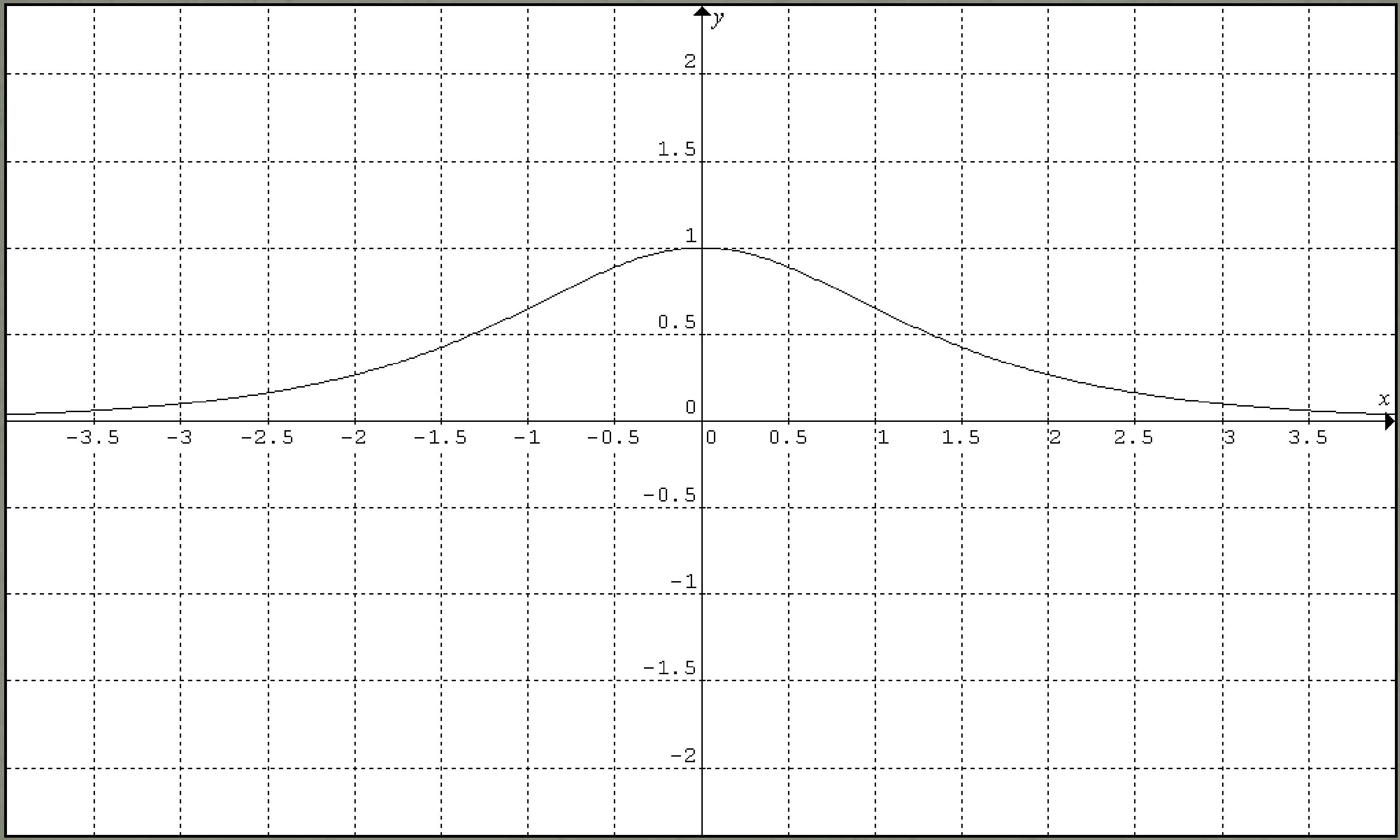
Range: (-∞, 0)∪(0,∞)



Graph of $f(x) = sech\ (x)$

Domain: (-∞, ∞)

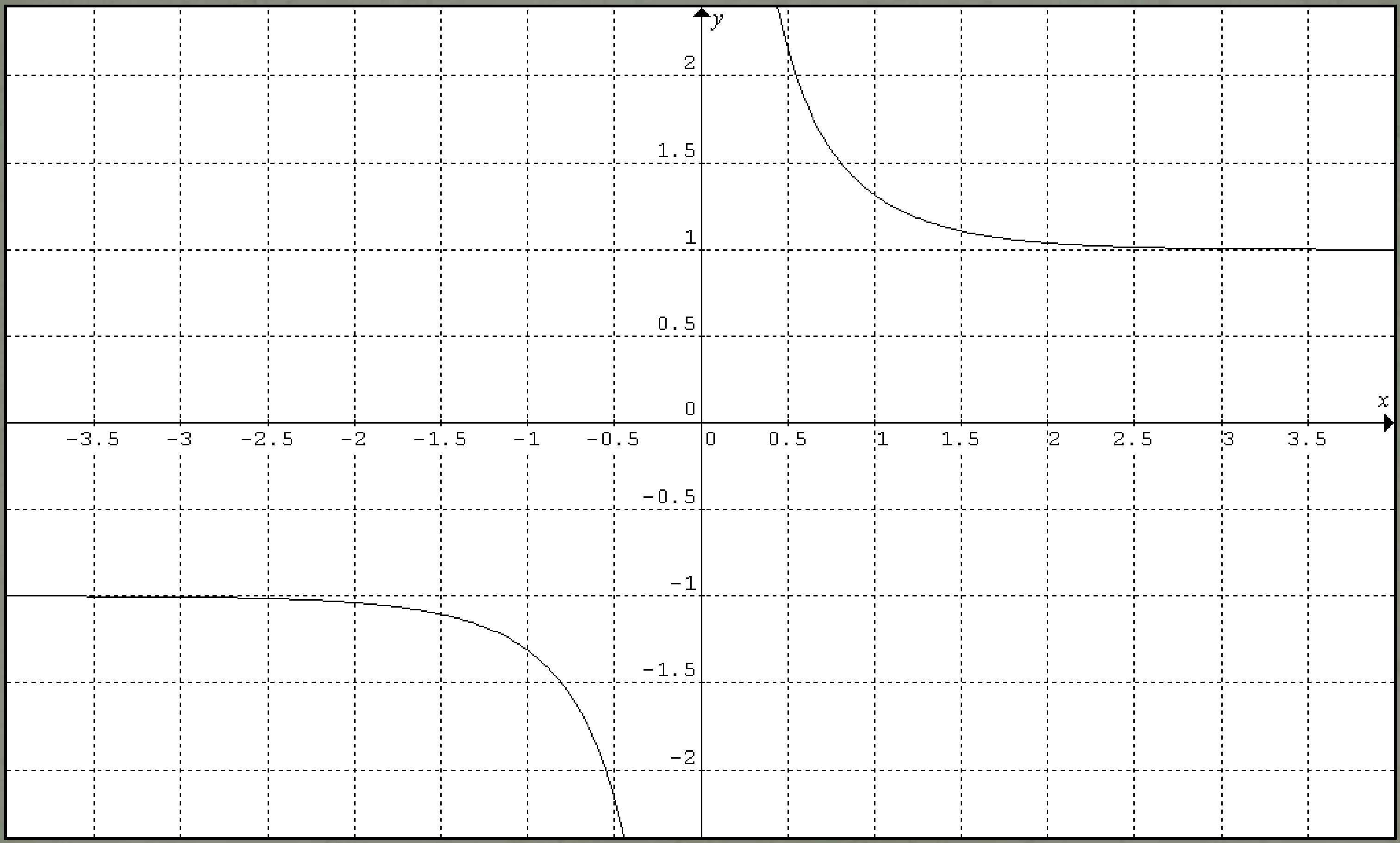
Range: (0, 1]



Graph of $f(x) = coth\ (x)$

Domain: $(-\infty, 0) \cup (0, \infty)$

Range: $(-\infty, -1) \cup (1, \infty)$



Hyperbolic Identities:

- $$cosh^2\text{x} - sinh^2x = 1$$
  $$tanh^2x + sech^2x = 1$$
  $$coth^2x - csch^2x = 1$$
  $$sinh^2x = \frac{-1 + \cosh 2x}{2}$$
  $$cosh^2x = \frac{1 + \cosh 2x}{2}$$
  $$\sinh 2x = 2 \sinh x \cdot cosh\ x$$
  $$\cosh 2x = cosh^2x + sinh^2x$$
  $$\sinh(x \pm y) = sinhx\ coshy \pm coshx\ sinhy$$
  $$\cosh(x \pm y) = coshx\ coshy \pm sinhx\ sinhy$$

Theorem 5.18: Derivatives and integrals of Hyperbolic functions.

- Let u be a differentiable function of x.

$$\frac{d}{dx}[\sinh u] = coshu \cdot u'$$

$$\frac{d}{dx}[\cosh u] = sinhu \cdot u'$$

$$\frac{d}{dx}[\tanh u] = sech^2 u \cdot u'$$

$$\frac{d}{dx}[\coth u] = -csch^2 u \cdot u'$$

$$\frac{d}{dx}[\operatorname{sech} u] = -(\operatorname{sech} u \cdot \tanh u) \cdot u'$$

$$\frac{d}{dx}[\operatorname{csch} u] = -(\operatorname{csch} u \cdot \coth u) \cdot u'$$

Theorem 5.18: Derivatives and integrals of Hyperbolic functions.

- $$\int \cosh u du = \sinh u + C$$

  $$\int \sinh u \, du = \cosh u + C$$

  $$\int sech^2 u \, du = \tanh u + C$$

  $$\int csch^2 u \, du = -\coth u + C$$

  $$\int sechu \cdot tanhu \, du = -\operatorname{sech} u + C$$

  $$\int cschu \cdot cothu \, du = -\operatorname{csch} u + C$$

Find the derivative of the function.

1. $f(x) = \cosh(x - 2)$

Find the derivative of the function.

1. $f(x) = \cosh(x - 2)$

   $f'(x) = 1 \cdot \sinh(x - 2)$

Find the derivative of the function.

2. $y = \tanh(3x^2 - 1)$

Find the derivative of the function.

2. $y = \tanh(3x^2 - 1)$

$y' = 6x \cdot \text{sech}^2 (3x^2 - 1)$

Find the derivative of the function.

3. $g(x) = x \cosh x - \sinh x$

Find the derivative of the function.

3. $g(x) = x \cosh x - \sinh x$

$$g'(x) = x \cdot \sinh x + \cosh x - \cosh x$$

$$g'(x) = x \cdot \sinh x$$

Find the integral

i. $\int \frac{cosh\sqrt{x}}{\sqrt{x}}dx$

Find the integral

- i. $\int \frac{cosh\sqrt{x}}{\sqrt{x}} dx$

  Let $u = \sqrt{x}$

  $du = \frac{1}{2} x^{-1/2} dx$

  $2du = \frac{dx}{\sqrt{x}}$

  Now substitute

  $\int \cosh u \cdot 2du$

  $= 2(\sinh u) + C$

  $= 2(\sinh \sqrt{x}) + C$

Find the integral

2. $\int \text{sech}^2 (2x - 1)\, dx$

Find the integral

2. $\int \text{sech}^2 (2x - 1)\, dx$

Let $u = 2x – 1$

$du = 2\, dx$

$du/2 = dx$

Now Substiture

$\int \text{sech}^2\, u\, du/2$

$= ½ \int \text{sech}^2\, u\, du$

$= ½ \tan u + C$

$= ½ \tan (2x – 1) + C$

Find the integral

- 3. $\int \frac{\cosh x}{\sqrt{9 - sinh^2 x}} dx$

Find the integral

- 3. $\int \frac{\cosh x}{\sqrt{9-sinh^2 x}} dx$

$let\ u^2 = \sinh^2 x$

$Then\ u = \sinh x$

$du = \cosh x\ dx$

$and\ a^2 = 9, thus\ a = 3$

Now Substitute

$\int \frac{du}{\sqrt{a^2-u^2}}$

$= \arcsin\left(\frac{u}{a}\right) + C$

$= \arcsin\left(\frac{\sinh x}{3}\right) + C$

Inverse Hyperbolic Functions

Function	Domain
$sinh^{-1}x = \ln(x + \sqrt{x^2+1})$	$(-\infty, \infty)$
$cosh^{-1}x = \ln\left(x + \sqrt{x^2-1}\right)$	$[1, \infty)$
$tanh^{-1}x = \frac{1}{2} ln\left(\frac{1+x}{1-x}\right)$	$(-1, 1)$
$coth^{-1}x = \frac{1}{2} ln\left(\frac{x+1}{x-1}\right)$	$(-\infty, -1) \cup (1, \infty)$
$sech^{-1}x = ln\left(\frac{1+\sqrt{1-x^2}}{x}\right)$	$(0, 1]$
$csch^{-1}x = \ln\left(\frac{1}{x} + \frac{\sqrt{1+x^2}}{\lvert x\rvert}\right)$	$(-\infty, 0) \cup (0, \infty)$

Differentiation involving inverse hyperbolic functions.

- $$\frac{d}{dx}[\sinh^{-1}u] = \frac{u'}{\sqrt{u^2+1}}$$
  $$\frac{d}{dx}[\cosh^{-1}u] = \frac{u'}{\sqrt{u^2-1}}$$
  $$\frac{d}{dx}[\tanh^{-1}u] = \frac{u'}{1-u^2}$$
  $$\frac{d}{dx}[\coth^{-1}u] = \frac{u'}{1-u^2}$$
  $$\frac{d}{dx}[\operatorname{sech}^{-1}u] = \frac{-u'}{u\sqrt{1-u^2}}$$
  $$\frac{d}{dx}[\operatorname{csch}^{-1}u] = \frac{-u'}{|u|\sqrt{1+u^2}}$$

Integration involving inverse hyperbolic functions.

- $$\int \frac{du}{\sqrt{u^2 \pm a^2}} = ln\left(u + \sqrt{u^2 \pm a^2}\right) + C$$

$$\int \frac{du}{a^2 - u^2} = \frac{1}{2a} ln \left|\frac{a+u}{a-u}\right| + C$$

$$\int \frac{du}{u\sqrt{a^2 \pm u^2}} = -\frac{1}{a} ln\left(\frac{a + \sqrt{a^2 \pm u^2}}{|u|}\right) + C$$