1.4 Functions

Definition: A function *f* from a set A to a set B is a relation that assigns to each element x in the set A exactly one element y in the set B. The set A is the domain (or set of inputs) of the function *f*, and the set B contains the range (or set of outputs).

Characteristics of a Function from Set A to Set B:

1. Each element in A must be matched with an element in B.

2. Some elements in B may not be matched with any element in A.

3. Two or more elements in A may be matched with the same element in B.

4. An element in A (the domain) cannot be matched with two different elements in B.

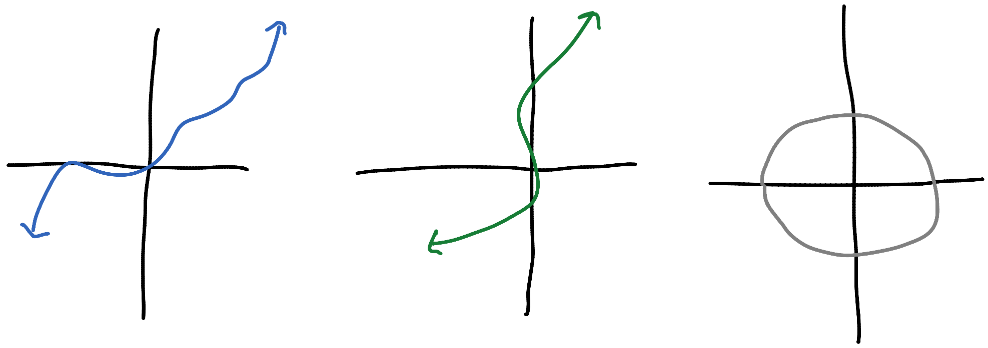
Four Ways to Represent a Function:

1. Verbally – by a sentence that describes how the input variable is related to the output variable.
2. Numerically –by a table or a list of ordered pairs that matches input values with output values.
3. Graphically – by points on a graph in a coordinate plane in which the input values are represented by the horizontal axis and the output values are represented by the vertical axis.
4. Algebraically – by an equation in two variables.

Frequently we refer to the domain values as the independent variable and the range values as the dependent variable. (This is because the output depends on the input we select.) As the input value is x and the output value is y, we can say that y depends on x. Or, using the word function, y is a function of x. But we would hate to write that out all the time so we use the notation $y = f(x)$ which means the same thing.

Examples: Function or not?

1. Some graphs.



yes, each x has at most one y

no, the value $x=0$ has multiple y values

no, many x's have two y values

2. A = { (2,3), (4,5), (-1, 3) }

even though both 2 and −1 go to 3, this is a function each x goes to only 1 y, a couple to the same one

B = { (1,-3), (2, 5), (1,7) }

each y value is different but $x=1$ goes to both $y=-3$ and $y=7$ This is <u>not</u> a function.

3. $x^2 + y^2 = 4$

a single x value, say $x=0$, has two possible y values:

$$0^2 + y^2 = 4 \quad \text{becomes} \quad y^2 = 4 \quad \text{or} \quad y = \pm 2.$$

This is <u>not</u> a function.

4. $y - 4x^2 = 36$

Any given x can only result in 1 y: $y = 4x^2 + 36$

so this is a function.

5. $x+y^2=16$

Once again any x-value gives two y results

try an easy $x=0$: $0+y^2=16$ so $y^2=16$ and $y=\pm 4$

Not a function

6. $y=\sqrt{x+5}$

Since y is already isolated it seems clear that any x substituted in the equation yields a single y. This is a function.

Cheaters Rule: If y has an even power, it is most likely not a function.

Examples: Evaluate the function and simplify.

1. $f(x)=2x-3; x=1, -3, x-1$

$f(1)=2(1)-3$
$=2-3$
$=-1$
so $f(1)=-1$

$f(-3)=2(-3)-3$
$=-6-3$
$=-9$
$f(-3)=-9$

$f(x-1)=2(x-1)-3$
$=2x-2-3$
$=2x-5$
$f(x-1)=2x-5$

2. $S(r) = 4\pi r^2; r = 2, 1/2, 3r$

$$S(2) = 4\pi(2)^2 = 4\pi(4) = 16\pi$$

exact!

$$S\left(\tfrac{1}{2}\right) = 4\pi\left(\tfrac{1}{2}\right)^2 = 4\pi\left(\tfrac{1}{4}\right) = \pi$$

$$S(3r) = 4\pi(3r)^2 = 4\pi(9r^2) = 36\pi r^2$$

3. $f(x) = \begin{cases} x^2 + 2, & x \le 1 \\ 2x^2 + 3, & x > 1 \end{cases}$; $x = -2, 1, 2$

Since $-2 \le 1$, we use the top piece: $f(-2) = (-2)^2 + 2 = 4 + 2 = 6$

Since $1 \le 1$ (equal), we again use the top piece: $f(1) = (1)^2 + 2 = 3$

Since $2 > 1$, we use the bottom piece: $f(2) = 2(2)^2 + 3 = 2(4) + 3 = 11$

Examples: Find all the real values of x such that *f(x) = 0*.

← Notice this is the x-intercept. Sometimes we call them zeros, or roots, of the function.

1. $f(x) = 5x + 1$

$$0 = 5x + 1$$
$$-1 = 5x$$
$$-\frac{1}{5} = x$$

2. $f(x) = \dfrac{12 - x^2}{5}$

$$5 \cdot 0 = \frac{12 - x^2}{\cancel{5}} \cdot \cancel{5}$$

$$0 = 12 - x^2$$
$$x^2 = 12$$
$$x = \pm\sqrt{12} = \pm\sqrt{4 \cdot 3} = \pm 2\sqrt{3}$$

Fact: When setting a rational expression equal to zero, we find where the numerator is zero.

3. $f(x)=x^3-x^2-4x+4$

$0 = x^3 - x^2 - 4x + 4$

$0 = x^2(x-1) - 4(x-1)$

$0 = (x-1)(x^2-4)$

$x-1=0$ or $x^2-4=0$

$x=1$ or $x^2=4$

$x = \pm\sqrt{4} = \pm 2$

$x = -2, 1, 2$

The domain of any function *f* consists of all real numbers x for which the function is defined. This means that we really only need to check denominators and radicands.

Denominators cannot be zero

Radicands, the part under the radical, of even index must be greater than or equal to zero.

Examples: Find the domain.

1. $g(x)=1-2x^2$ No denominator, no radical

so Domain is all real numbers $= (-\infty, \infty)$

2. $s(y)=\dfrac{3y}{y+5}$ Denominator cannot be zero so

$y+5 \neq 0$ or $y \neq -5$

Domain $y \neq -5$

In general our domain will refer to x. However for this function y is the input so it tells us domain.

$\boxed{(-\infty,-5) \cup (-5,\infty)}$

3. $f(x)=\dfrac{\sqrt{x+6}}{6+x}$

Denominator cannot be zero: $6+x \neq 0$ so $x \neq -6$

Radicand must be ≥ 0: $x+6 \geq 0$ so $x \geq -6$.

Now $x \geq -6$ But $x \neq -6$ so the domain is $x > -6$ or $(-6, \infty)$

Examples:

1. Write the area A of a square as a function of its perimeter P.

$A = s^2$ side length s $P = 4s$

Area is in terms of s so rewrite perimeter as $s = \frac{P}{4}$

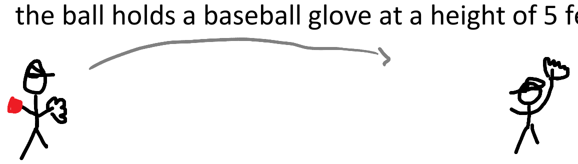
to get $A = s^2 = \left(\frac{P}{4}\right)^2$ or $A = \frac{P^2}{16}$

2. Write the area A of a circle as a function of its circumference C.

$A = \pi r^2$ $C = 2\pi r \rightarrow r = \frac{C}{2\pi}$

$A = \pi\left(\frac{C}{2\pi}\right)^2 = \pi \frac{C^2}{4\pi^2} = \frac{C^2}{4\pi}$. That is, $A(C) = \frac{C^2}{4\pi}$

3. The height y (in feet) of a baseball thrown by a child is $y=-\frac{1}{10}x^2+3x+6$ where x is the horizontal distance (in feet) from where the ball was thrown. Will the ball fly over the head of another child 30 feet away trying to catch the ball? (Assume that the child who is trying to catch the ball holds a baseball glove at a height of 5 feet.)



height is y so find y when $x = 30$ ft.

$$y = -\frac{1}{10}(30)^2 + 3(30) + 6$$
$$= -\frac{1}{10}(900) + 90 + 6$$
$$= -90 + 90 + 6$$
$$= 6$$

After the ball travels 30 ft it will be 6 ft in the air. The other kid will not catch it.

One of the basic definitions in calculus uses the difference quotient:

$$\frac{f(x+h)-f(x)}{h},\ h \neq 0.$$

This can also be given as $\frac{f(t)-f(c)}{t-c},\ t \neq c$

Examples: Find the difference quotient and simplify.

1. $f(x)=5x-x^2,\ \frac{f(5+h)-f(5)}{h},\ h \neq 0$

$$f(5+h) = 5(5+h)-(5+h)^2$$
$$= 25+5h-(25+10h+h^2)$$
$$= 25+5h-25-10h-h^2$$
$$= -5h-h^2$$

$$f(5) = 5(5)-(5)^2$$
$$= 25-25$$
$$= 0$$

$$\frac{f(5+h)-f(5)}{h} = \frac{-5h-h^2-0}{h} = \frac{-5h-h^2}{h} = \frac{\cancel{h}(-5-h)}{\cancel{h}} = -5-h$$

2. $f(t)=\frac{1}{t-2},\ \frac{f(t)-f(1)}{t-1},\ t \neq 1$

$$f(1) = \frac{1}{1-2} = \frac{1}{-1} = -1$$

$$f(t)-f(1) = \frac{1}{t-2} - -1 = \frac{1}{t-2} + 1 \cdot \frac{t-2}{t-2} = \frac{1}{t-2} + \frac{t-2}{t-2} = \frac{t-1}{t-2}$$

$$\frac{f(t)-f(1)}{t-1} = \frac{\frac{t-1}{t-2}}{t-1} = \frac{t-1}{t-2} \cdot \frac{1}{t-1} = \frac{1}{t-2}$$

Notes: ① Dividing by $t-1$ is the same as multiplying by $\frac{1}{t-1}$

② Do not expect the original function back every time.