1.8 Combinations of Functions: Composite Functions

Sum, Difference, Product, and Quotient of Functions – Let *f* and *g* be two functions with overlapping domains. Then, for all *x* common to both domains, the sum, difference, product, and quotient of *f* and *g* are defined as follows.

1. Sum: $(f+g)(x) = f(x) + g(x)$

2. Difference: $(f-g)(x) = f(x) - g(x)$

3. Product: $(fg)(x) = f(x)g(x)$

4. Quotient: $\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)},\ g(x) \neq 0$

Example: Find the sum, difference, product and quotient for $f(x) = 3x+1$ and $g(x) = \sqrt{1-x}$.

$(f+g)(x) = f(x) + g(x) = 3x+1+\sqrt{1-x}$

$(f-g)(x) = f(x) - g(x) = 3x+1-\sqrt{1-x}$

$(fg)(x) = f(x) \cdot g(x) = (3x+1)\sqrt{1-x}$

$\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)} = \dfrac{3x+1}{\sqrt{1-x}}, \quad x < 1$

All of those act just like we would expect them too. The fifth operation on functions is the important one because it is not found when combining other mathematical objects.

$(f \circ g)(x) = f(g(x))$

Definition: The composition of the function *f* with the function *g* is $(f \circ$ $f(g(x))$. The domain of $f \circ$ $f \circ g$ is the set of all *x* in the domain of *g* such that *g(x)* is in the domain of *f*.

In $(f \circ g)(x) = f(g(x))$, I will refer to $g(x)$ as the "inside" function and $f(x)$ as the "outside" function.

Examples: Find the compositions in both orders for the given functions.

1. $f(x)=\sqrt[3]{x-5}$ and $g(x)=x^3+1$

$$(f\circ g)(x)=f(g(x))$$
$$=f(x^3+1)$$
$$=\sqrt[3]{(x^3+1)-5}$$
$$=\sqrt[3]{x^3-4}$$

$$(g\circ f)(x)=g(f(x))$$
$$=g(\sqrt[3]{x-5})$$
$$=(\sqrt[3]{x-5})^3+1$$
$$=x-5+1$$
$$=x-4$$

*Notice $f\circ g \neq g\circ f$ in general *

2. $f(x)=|x-4|$ and $g(x)=3-x$

$$(f\circ g)(x)=f(g(x))$$
$$=f(3-x)$$
$$=|(3-x)-4|$$
$$=|-1-x|$$

$$(g\circ f)(x)=g(f(x))$$
$$=g(|x-4|)$$
$$=3-|x-4|$$

$(f\circ g)(x)=h(x)$

Examples: Find two functions *f* and *g* such that $(f\circ g)(x)=h(x)$ Answers vary greatly!!

This is a valuable skill for Calculus.

1. $h(x)=(1-x)^3$

in $=1-x$ so let $g(x)=1-x$

out $=(\ )^3$ $\quad f(x)=x^3$

2. $h(x)=\sqrt{9-x}$

in $=9-x$ so let $g(x)=9-x$

out $=\sqrt{(\ )}$ $\quad f(x)=\sqrt{x}$

3. $h(x)=\dfrac{4}{(5x+2)^2}$

One possibility is

$g(x)=5x+2$

$f(x)=\dfrac{4}{x^2}$

another possibility is $g(x)=(5x+2)^2$

$f(x)=\dfrac{4}{x}$

But there are many other options as well.

4. $h(x)=\dfrac{27x^3+6x}{10-27x^3}$

This one is a challenge. Notice that $(3x)^3=27x^3$

so...

let $g(x)=3x$

and $f(x)=\dfrac{x^3+2x}{10-x^3}$

5. $h(x)=\dfrac{1}{x+2}$

Let $g(x)=x+2$ and $f(x)=\dfrac{1}{x}$

6. $h(x)=\sqrt[3]{x^2-4}$

Let $g(x)=x^2-4$ and $f(x)=\sqrt[3]{x}$

Example: The number *N* of bacteria in a refrigerated food is given by

$N(T)=10T^2-20T+600,\ 1\le T\le 20$ where *T* is the temperature of the food in degrees Celsius.

When the food is removed from refrigeration, the temperature of the food is given by

$T(t)=3t+2,\ 0\le t\le 6$ where *t* is time in hours.

a) Find the composition *N(T(t))* and interpret its meaning in context.

$$\begin{aligned}N(T(t)) = N(3t+2) &= 10(3t+2)^2-20(3t+2)+600\\ &= 10(9t^2+12t+4)-60t-40+600\\ &= 90t^2+120t+40-60t+560\\ &= 90t^2+60t+600\end{aligned}$$

b) Find the bacteria count after 0.5 hours.

$$N(T(0.5)) = 90(.5)^2 + 60(.5) + 600$$
$$= 90(.25) + 30 + 600$$
$$= 22.5 + 30 + 600$$
$$= 652.5 \text{ bacteria}$$

c) Find the $\underbrace{\text{time}}_{t}$ when the bacteria $\underbrace{\text{count reaches 1500}}_{N=1500}$.

$$1500 = 90t^2 + 60t + 600$$
$$0 = 90t^2 + 60t - 900 \qquad \text{Divide by 30}$$
$$0 = 3t^2 + 2t - 30$$
$$t = \frac{-(2) \pm \sqrt{(2)^2 - 4(3)(-30)}}{2(3)} = \frac{-2 \pm \sqrt{364}}{6}$$

$t \approx -3.5$ hours or $t = 2.846$ hours

about 2.8 hours or 2h 48min.