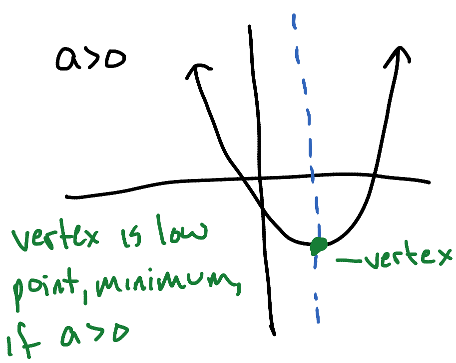
Chapter Two – Polynomial and Rational Functions

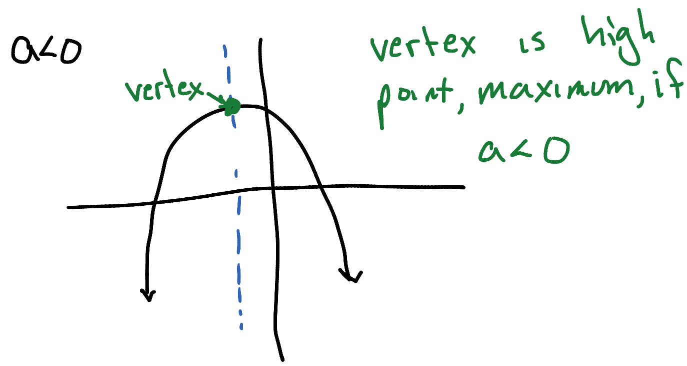
2.1 Quadratic Functions and Models

Definition of Polynomial Function – Let *n* be a nonnegative integer and let $a_n, a_{n-1}, \ldots, a_2, a_1, a_0$ be real numbers with $a_n \neq 0$. The function given by $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$ is called a polynomial function of *x* with degree *n*.

Definition of a Quadratic Function – Let *a, b,* and *c* be real numbers with $a \neq 0$. The function given by $f(x) = ax^2 + bx + c$ is called a quadratic function.

The graph of every quadratic function has the shape of a parabola. All parabolas are symmetric with respect to a line called the axis of symmetry, or simply the axis. The point where the axis intersects the parabola is called the vertex, or turning point, of the graph. When $a > 0$ the parabola opens upward and when $a < 0$ the parabola opens downward.





Standard Form of a Quadratic Function – The quadratic function given by $f(x) = a(x-h)^2 + k$ is in standard form. The graph of *f* is a parabola whose axis is the vertical line *x= h* and whose vertex is the point *(h,k)*.

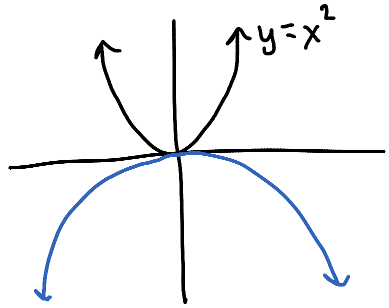
Notice: $f(x) = a(x-h)^2 + k$ moves left/right h units, up/down k units, and has a vertical stretch/shrink by a units.

Examples: Graph each function and compare with the graph of $y = x^2$.

1. $g(x) = -\frac{1}{8}x^2$

reflect

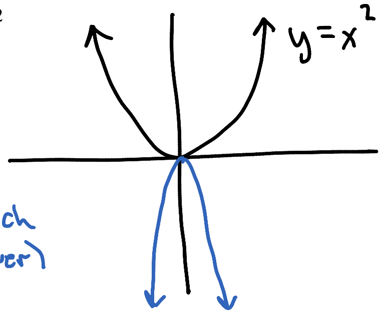
vertical shrink by $\frac{1}{8}$ (wider)



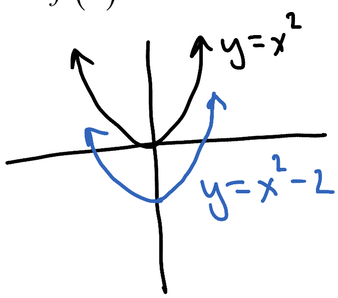
2. $h(x) = -3x^2$

reflect

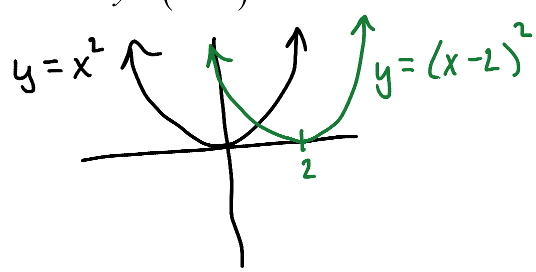
vert. stretch (narrower)



3. $f(x) = x^2 - 2$



4. $y = (x-2)^2$

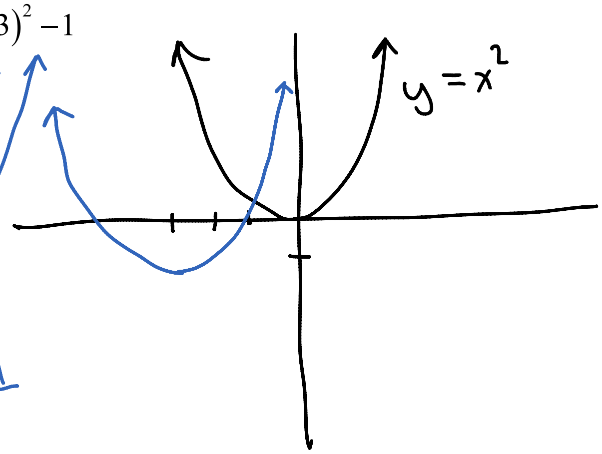


5. $f(x) = \frac{1}{2}(x+3)^2 - 1$

wider

left 3

down 1



The definition of the quadratic and the standard form of a quadratic are not the same form. The standard form allows you to look and see the vertex and from there it is somewhat easy to graph. Can the same thing be accomplished with the form from the definition? Why yes it can!

Minimum and Maximum Values of Quadratic Functions – Consider the function $f(x) = ax^2 + bx + c$ with vertex $\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$.

1. If $a > 0$, f has a minimum at $x = \frac{-b}{2a}$. The minimum value is $f\left(\frac{-b}{2a}\right)$.

2. If $a < 0$, f has a maximum at $x = \frac{-b}{2a}$. The maximum value is $f\left(\frac{-b}{2a}\right)$.

Examples: Sketch the graph of the quadratic function. Identify the vertex, axis of symmetry, and x-intercepts.

1. $f(x) = -x^2 - x + 30$ $a=-1, b=-1, c=30$

vertex: $x = \frac{-(-1)}{2(-1)} = \frac{1}{-2} = -0.5$

$(-0.5, 30.25)$ $y = f(-\frac{1}{2}) = 30.25$

$x = -0.5$

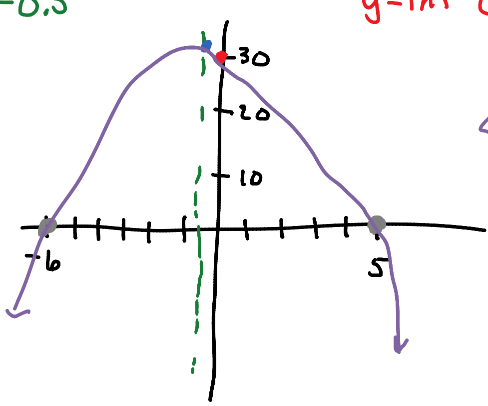
y-int $(0,30)$

$0 = -x^2 - x + 30$

$x^2 + x - 30 = 0$

$(x+6)(x-5) = 0$

$x = -6, x = 5$ or $(-6,0)$ & $(5,0)$



← ugly but mostly accurate :)

2. $g(x) = -4x^2 + 24x - 41$ $a=-4,\ b=24,\ c=-41$

vertex: $x = \dfrac{-(24)}{2(-4)} = \dfrac{-24}{-8} = 3$

$(3,-5)$ $y = g(3) = -5$

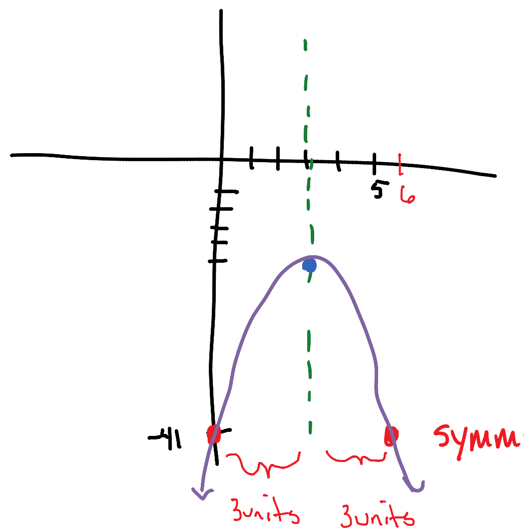
$x = 3$

y-int $(0,-41)$

$0 = -4x^2 + 24x - 41$ factor? no way!

$x = \dfrac{-(24) \pm \sqrt{(24)^2 - 4(-4)(-41)}}{2(-4)}$

$x = \dfrac{-24 \pm \sqrt{-80}}{-8}$ ← no solution so no x-int.



symmetric point to y-intercept

equidistant from axis of symmetry to y-int on opposite side with same height as y-int.

3. $y = 2x^2 - 16x + 31$ $a=2,\ b=-16,\ c=31$

vertex: $x = \dfrac{-(-16)}{2(2)} = \dfrac{16}{4} = 4$

$(4,-1)$ $y = 2(4)^2 - 16(4) + 31 = -1$

$x = 4$

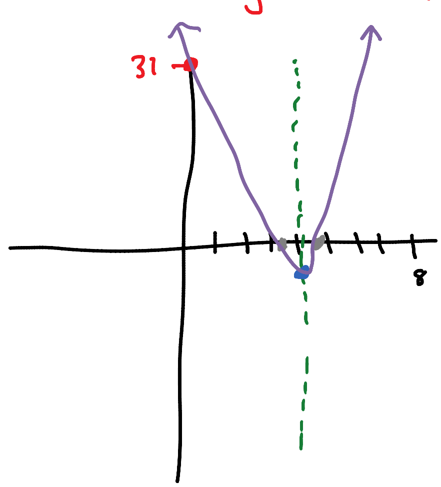
y-int $(0,31)$

$0 = 2x^2 - 16x + 31$

$x = \dfrac{-(-16) \pm \sqrt{(-16)^2 - 4(2)(31)}}{2(2)}$

$x = \dfrac{16 \pm \sqrt{8}}{4} = \dfrac{16 \pm 2\sqrt{2}}{4}$

$x = \dfrac{16 + 2\sqrt{2}}{4} \approx 4.707$ $x = \dfrac{16 - 2\sqrt{2}}{4} \approx 3.29$



Examples: Write the standard form of the equation of the parabola that has the indicated vertex and whose graph passes through the given point. Standard form: $y = a(x-h)^2 + k$

1. Vertex: $(\underset{h}{4}, \underset{k}{-1})$; point: $(\underset{x}{2}, \underset{y}{3})$ to find a

$y = a(x-4)^2 - 1$

$3 = a(2-4)^2 - 1$

$3 = a(-2)^2 - 1$

$4 = 4a$

$1 = a$

with $a = 1$, $h = 4$, $k = -1$ we have

$y = 1(x-4)^2 - 1 = (x-4)^2 - 1$

2. Vertex: (-2, -2); point: (-1, 0)

$y = a(x-(-2))^2 - 2$

$y = a(x+2)^2 - 2$

$0 = a(-1+2)^2 - 2$

$0 = a(1)^2 - 2$

$2 = a$

$y = 2(x+2)^2 - 2$

3. You try it: Vertex: (-3, 5); point: (1, 9)

Example: The path of a diver is given by $y = -\frac{4}{9}x^2 + \frac{24}{9}x + 12$ where *y* is the height (in feet) and *x* is the horizontal distance from the end of the diving board (in feet). What is the maximum height of the diver?

vertex

$x = \frac{-b}{2a} = \frac{-\left(\frac{24}{9}\right)}{2\left(-\frac{4}{9}\right)} = \frac{-\frac{24}{9}}{-\frac{8}{9}} = \frac{-24}{\cancel{9}} \cdot \frac{\cancel{9}}{-8} = \frac{-24}{-8} = 3$ ft from end of board

$y = -\frac{4}{9}(3)^2 + \frac{24}{9}(3) + 12 = -\frac{4}{\cancel{9}}(\cancel{9}) + \frac{24}{\cancel{9}_{3}}(\cancel{3}) + 12$

$= -4 + 8 + 12 = 16$ ft max height

Example: A manufacturer of lighting fixtures has daily production costs of $C = 800 - 10x + 0.25x^2$, where C is the total cost (in dollars) and x is the number of units produced. How many fixtures should be produced each day to yield minimum cost?

Find x

vertex

Rewrite equation: $C = 0.25x^2 - 10x + 800$

vertex is at $x = \dfrac{-(-10)}{2(0.25)} = \dfrac{10}{0.5} = 20$ fixtures per day

Bonus: the minimum cost would be $C(20) = \$700$