2.3 Polynomial and Synthetic Division

The Division Algorithm – If *f(x)* and *d(x)* are polynomials such that *d(x)≠0,* and the degree of *d(x)* is less than or equal to the degree of *f(x)*, there exists unique polynomials *q(x)* and *r(x)* such that $f(x)=d(x)q(x)+r(x)$ where *r(x) = 0* or the degree of *r(x)* is less than the degree of *d(x)*. If the remainder is zero, *d(x)* divides evenly into *f(x)*.

Remember how to divide by hand:

$$\begin{array}{r} 38 \\ 13\overline{)496} \\ -39\phantom{6} \\ \hline 106 \\ -104 \\ \hline 2 \end{array}$$

So $\frac{496}{13} = 38\,R2$ or $38+\frac{2}{13} = 38\frac{2}{13}$.

$496 = 13(38)+2$

Examples: Use long division to divide.

1. $(\underbrace{5x^2-17x-12}_{f(x)})\div(\underbrace{x-4}_{d(x)})$

$$\begin{array}{r} 5x+3 \quad \leftarrow q(x) \\ x-4\overline{)5x^2-17x-12} \\ -(5x^2-20x)\phantom{-12} \\ \hline 3x-12 \\ -(3x-12) \\ \hline 0 \quad \leftarrow r(x) \end{array}$$

$\frac{5x^2-17x-12}{x-4} = 5x+3$

or using division algorithm

$5x^2-17x-12 = (x-4)(5x+3)+0$

$f(x) = d(x)\cdot q(x)+r(x)$

2. $(6x^3-16x^2+17x-6)\div(3x-2)$

$$\begin{array}{r} 2x^2-4x+3 \\ 3x-2\overline{)6x^3-16x^2+17x-6} \\ -(6x^3-4x^2)\phantom{+17x-6} \\ \hline -12x^2+17x\phantom{-6} \\ -(-12x^2+8x)\phantom{-6} \\ \hline 9x-6 \\ -(9x-6) \\ \hline 0 \end{array}$$

So

$6x^3-16x^2+17x-6 = (3x+2)(2x^2-4x+3)$

3. $\left(x^3+125\right)\div(x+5)$

$$
\begin{array}{r|l}
 & \quad x^2 - 5x + 25 \\
\hline
x+5 & x^3 + 0x^2 + 0x + 125 \\
 & -\left(x^3 + 5x^2\right) \\
\hline
 & \qquad -5x^2 + 0x \\
 & \qquad -\left(-5x^2 - 25x\right) \\
\hline
 & \qquad\qquad 25x + 125 \\
 & \qquad\qquad -\left(25x + 125\right) \\
\hline
 & \qquad\qquad\qquad 0
\end{array}
$$

$$x^3+125=(x+5)(x^2-5x+25)$$

Synthetic Division (for a Cubic Polynomial) – To divide $ax^3+bx^2+cx+d$ by $x-k$, use the following pattern.

a drops down

$$
\begin{array}{c|cccc}
k & a & b & c & d \\
 & & ka & & \\
\hline
 & a & b+ka & & r
\end{array}
$$

multiply by k to move up and over

add straight down

The vertical pattern is to add terms and the diagonal pattern is to multiply by *k*.

Examples: Use synthetic division to divide.

1. $\left(3x^3-17x^2+15x-25\right)\div\underline{(x-5)}$

$$
\begin{array}{c|cccc}
5 & 3 & -17 & 15 & -25 \\
 & & 15 & -10 & 25 \\
\hline
 & 3 & -2 & 5 & 0
\end{array}
$$

$\times5$ $\quad +$

$r(x)=0$

$q(x)=3x^2-2x+5$

$q(x)$ is always one degree less than $f(x)$ in synthetic division.

2. $\left(5x^3+18x^2+7x-6\right)\div\left(x+3\right)$

$$\begin{array}{r|rrrr} -3 & 5 & 18 & 7 & -6 \\ & & -15 & -9 & 6 \\ \hline & 5 & 3 & -2 & 0 \end{array}$$

$q(x) = 5x^2+3x-2$

$r(x) = 0$

3. $\left(-x^3+75x-250\right)\div\left(x+10\right)$

$$\begin{array}{r|rrrr} -10 & -1 & 0 & 75 & -250 \\ & & 10 & -100 & 250 \\ \hline & -1 & 10 & -25 & 0 \end{array}$$

$q(x) = -x^2+10x-25$

$r(x) = 0$

4. You try it: $\left(x^3-7x^2+9\right)\div\left(x-2\right)$ Note: The remainder will not always be 0.

The Remainder Theorem – If a polynomial *f(x)* is divided by *x – k*, the remainder is $r = f\left(k\right)$ .

The Factor Theorem – A polynomial *f(x)* has a factor *(x – k)* if and only if $f\left(k\right)=0$ .

Using the Remainder in Synthetic Division – The remainder *r*, obtained in the synthetic division of *f(x)* by *x – k*, provides the following information.

1. The remainder gives the value of *f* at *x = k*.

2. If the remainder is zero, *(x – k)* is a factor of *f*.

3. If *r = 0, (k 0)* is an *x*-intercept of the graph of *f*.

Example: Write $f(x)=x^3-5x^2-11x+8$ in the form of $f(x)=(x-4)q(x)+r$ and demonstrate that $f(k)=r$.

asking us to divide by x-4

$$
\begin{array}{c|cccc}
 & x^3 & x^2 & x & \text{const} \\
4 & 1 & -5 & -11 & 8 \\
 & & 4 & -4 & -60 \\
\hline
 & 1 & -1 & -15 & -52 \\
 & x^2 & x & \text{const} & \text{rem}
\end{array}
$$

$r=-52$

$f(x)=(x-4)(x^2-x-15)-52$

$$
\begin{aligned}
f(4) &= (4)^3-5(4)^2-11(4)+8 \\
&= 64-5(16)-44+8 \\
&= 64-80-44+8 \\
&= -124+72 \\
&= -52
\end{aligned}
$$

Examples: Use the Remainder Theorem and synthetic division to find each function value for $g(x)=2x^6+3x^4-x^2+3$.

1. *g(2)*

$$
\begin{array}{c|ccccccc}
2 & 2 & 0 & 3 & 0 & -1 & 0 & 3 \\
 & & 4 & 8 & 22 & 44 & 86 & 172 \\
\hline
 & 2 & 4 & 11 & 22 & 43 & 86 & 175
\end{array}
$$

2. *g(-1)*

$$
\begin{array}{c|ccccccc}
-1 & 2 & 0 & 3 & 0 & -1 & 0 & 3 \\
 & & -2 & 2 & -5 & 5 & -4 & 4 \\
\hline
 & 2 & -2 & 5 & -5 & 4 & -4 & 7
\end{array}
$$