2.4 Complex Numbers

For centuries mathematics has been an ever-expanding field because of one particular "trick." Whenever a notable mathematician gets stuck on a problem that seems to have no solution, they make up something new. This is how complex numbers were "invented." A simple quadratic equation would be $x^2+1=0$. However, in trying to solve this it was found that $x^2=-1$ and that was confusing. How could a quantity multiplied by itself equal a negative number?

This is where the genius came in. A guy named Cardano developed complex numbers off the base of the imaginary number $i=\sqrt{-1}$, the solution to our "easy" equation $x^2=-1$. The system didn't really get rolling until Euler and Gauss started using it, but if you want to blame someone it should be Cardano.

My Definition – The imaginary unit $i$ is a number such that $i^2=-1$. That is, $i=\sqrt{-1}$.

Definition of a Complex Number – If *a* and *b* are real numbers, the number *a + bi* is a complex number, and it is said to be written in standard form. If *b = 0*, the number *a + bi = a* is a real number. If $b \neq 0$, the number *a + bi* is called an imaginary number. A number of the form *bi*, where $b \neq 0$, is called a pure imaginary number.

Equality of Complex Numbers – Two complex numbers *a + bi* and *c + di*, written in standard form, are equal to each other $a+bi=c+di$ if and only if *a = c* and *b = d*.

Addition and Subtraction of Complex Numbers – If *a + bi* and *c + di* are two complex numbers written in standard form, their sum and difference are defined as follows.

Sum: $$(a+bi)+(c+di)=(a+c)+(b+d)i$$

Difference: $$(a+bi)-(c+di)=(a-c)+(b-d)i$$

To multiply complex numbers, use the distributive property keeping in mind that $i^2=-1$.

Examples: Write the complex number in standard form.

1. $5+\sqrt{-36}$

$5+\sqrt{36(-1)} = 5+6i$

$\sqrt{36(-1)} = \sqrt{36}\sqrt{-1} = 6i$

2. $1+\sqrt{-8}$

$1+\sqrt{8(-1)} = 1+\sqrt{4(-1)(2)} = 1+2i\sqrt{2}$

$\sqrt{4(-1)(2)} = \sqrt{4}\sqrt{-1}\sqrt{2} = 2i\sqrt{2}$

3. $-4i^2+2i$

$-4(-1)+2i = 4+2i$

4. $\sqrt{-4}$

$\sqrt{4(-1)} = \sqrt{4}\sqrt{-1} = 2i$

5. You try it: $\sqrt{-75}+3i^2$

Examples: Perform the addition or subtraction and write the result in standard form.

1. $(13-2i)+(-5+6i)$

$= (13+(-5)) + (-2i+6i) = 8+4i$

2. $(3+2i)-(6+13i)$

$= (3-6)+(2i-13i) = -3-11i$

3. $\left(-2+\sqrt{-8}\right)+\left(5-\sqrt{-50}\right)$ $= \left(-2+2i\sqrt{2}\right)+\left(5-5i\sqrt{2}\right) = 3-3i\sqrt{2}$

$\sqrt{-8} = \sqrt{4(-1)(2)} = 2i\sqrt{2}$

$\sqrt{-50} = \sqrt{25(-1)(2)} = 5i\sqrt{2}$

Examples: Multiply and write the result in standard form.

1. $(7-2i)(3-5i)$

$= 21-35i-6i+10i^2$

$= 21-41i+10(-1)$

$= 21-41i-10$

$= 11-41i$

2. $-8i(9+4i)$

$= -72i-32i^2$

$= -72i-32(-1)$

$= -72i+32$ or $32-72i$

3. $\left(\sqrt{3}+\sqrt{15}i\right)\left(\sqrt{3}-\sqrt{15}i\right)$

$= \sqrt{3}\sqrt{3}-\sqrt{3}\sqrt{15}i+\sqrt{15}i\sqrt{3}-(\sqrt{15}i)(\sqrt{15}i)$

$= 3-15i^2$

$= 3-15(-1)$

$= 3+15 = 18$

When factoring, we have a formula called the difference of squares: $a^2 - b^2 = (a+b)(a-b)$. The factors on the right side of the equation are known as conjugates. In this section we are concerned with complex conjugates and have a new factoring/multiplying formula: $(a+bi)(a-bi) = a^2 + b^2$. We use complex conjugates to "rationalize" the denominators of quotients involving complex numbers.

Examples: Write the quotient in standard form.

1. $\frac{-14}{2i} = \frac{-7}{i} \cdot \frac{i}{i} = \frac{-7i}{i^2} = \frac{-7i}{-1} = 7i$

2. $\frac{13}{1-i} \cdot \frac{1+i}{1+i} = \frac{13+13i}{1^2+1^2} = \frac{13+13i}{2} = \frac{13}{2} + \frac{13}{2}i$

3. $\frac{6-7i}{1-2i} \cdot \frac{1+2i}{1+2i} = \frac{6+12i-7i-14i^2}{1^2+2^2} = \frac{6+5i+14}{5} = \frac{20+5i}{5} = 4+i$

4. You try it: $\frac{2-i}{1+7i}$

Examples: Perform the operation and write the result in standard form.

1. $\frac{2i}{2+i} + \frac{5}{2-i}$

To add fractions we need a common denominator.

$\frac{2i}{2+i} \cdot \frac{2-i}{2-i} = \frac{4i-2i^2}{2^2+1^2} = \frac{2+4i}{5}$

$\frac{5}{2-i} \cdot \frac{2+i}{2+i} = \frac{10+5i}{2^2+1^2} = \frac{10+5i}{5}$

we could simplify, instead we keep the common denom. in order to add

$\frac{2+4i}{5} + \frac{10+5i}{5} = \frac{12+9i}{5} = \boxed{\frac{12}{5} + \frac{9}{5}i}$

2. $\dfrac{1+i}{i} - \dfrac{3}{4-i}$

$$\frac{1+i}{i} \cdot \frac{i}{i} = \frac{i+i^2}{i^2} = \frac{-1+i}{-1} = 1-i$$

$$\frac{3}{4-i} \cdot \frac{4+i}{4+i} = \frac{12+3i}{4^2+1^2} = \frac{12+3i}{17}$$

$$1-i-\frac{12+3i}{17} = \frac{1-i}{1} \cdot \frac{17}{17} - \frac{12+3i}{17} = \frac{17-17i}{17} - \frac{12+3i}{17} = \frac{5-20i}{17}$$

$$= \frac{5}{17} - \frac{20}{17} i$$

Principal Square Root of a Negative Number – If *a* is a positive number, the principal square root of the negative number –*a* is defined as $\sqrt{-a} = \sqrt{a}i = i\sqrt{a}$.

My Definition – The solutions of $ax^2+bx+c=0$ are given by the quadratic formula to be $x = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$. When you simplify, simplify the radical first and then the overall fraction.

Examples: Use the Quadratic Formula to solve the quadratic equation.

1. $x^2+6x+10=0$

$a=1 \quad b=6 \quad c=10$

$$x = \frac{-(6) \pm \sqrt{(6)^2-4(1)(10)}}{2(1)}$$

$$x = \frac{-6 \pm \sqrt{-4}}{2}$$

$$x = \frac{-6 \pm 2i}{2}$$

$$x = -3 \pm i$$

2. $16t^2-4t+3=0$

$a=16 \quad b=-4 \quad c=3$

$$t = \frac{-(-4) \pm \sqrt{(-4)^2-4(16)(3)}}{2(16)}$$

$$t = \frac{4 \pm \sqrt{-176}}{32} = \frac{4 \pm \sqrt{16(-1)(11)}}{32}$$

$$t = \frac{4 \pm 4i\sqrt{11}}{32} = \frac{4}{32} \pm \frac{4i\sqrt{11}}{32}$$

$$t = \frac{1}{8} \pm \frac{\sqrt{11}}{8} i$$