2.5 Zeros of Polynomial Functions

The Fundamental Theorem of Algebra – If *f(x)* is a polynomial of degree *n*, where *n > 0*, then *f* has at least one zero in the complex number system.

Linear Factorization Theorem – If *f(x)* is a polynomial of degree *n,* where *n > 0*, then *f* has precisely *n* linear factors $f(x)=a_n(x-c_1)(x-c_2)\ldots(x-c_n)$ where $c_1, c_2, \ldots, c_n$ are complex numbers.

Examples: Find all the zeros of the function. Zeros = x-intercepts = Roots are when $y=0$.

1. $f(x)=x^2(x+3)(x^2-1)$

   $0 = x^2(x+3)(x^2-1)$

   $x^2=0$ or $x+3=0$ or $x^2-1=0$

   $x=0$ or $x=-3$ or $x^2=1$

   $x=\pm 1$

2. $f(x)=(x+6)(x+i)(x-i)$

   $0=(x+6)(x+i)(x-i)$

   $x+6=0$ or $x+i=0$ or $x-i=0$

   $x=-6$ or $x=-i$ or $x=i$

Factors and zeros are closely related!

The Rational Zero Test – If the polynomial $f(x)=a_nx^n+a_{n-1}x^{n-1}+\ldots+a_2x^2+a_1x+a_0$ has integer coefficients, every rational zero of *f* has the form Rational Zero = $\frac{p}{q}$ where *p* and *q* have no common factors other than 1, and *p* is a factor of a_0 and *q* is a factor of a_n .

Examples: Find all the rational zeros of the function.

1. $f(x) = x^3 - 7x - 6$

all possible rational zeros are

$$\frac{p}{q} = \frac{\pm(1,2,3,6)}{\pm 1} = \pm(1,2,3,6)$$

To find actual zeros either use synthetic division or evaluate.

$$\begin{array}{r|rrrr} 1 & 1 & 0 & -7 & -6 \\ & & 1 & 1 & -6 \\ \hline & 1 & 1 & -6 & -12 \end{array}$$

not a zero

$$\begin{array}{r|rrrr} -1 & 1 & 0 & -7 & -6 \\ & & -1 & 1 & 6 \\ \hline & 1 & -1 & -6 & 0 \end{array}$$

so $x=-1$ is a zero

rewrite $x^3 - 7x - 6 = (x+1)(x^2 - x - 6)$

Now we can factor the quadratic $x^2 - x - 6 = (x-3)(x+2)$

To get factors

$x^3 - 7x - 6 = (x+1)(x-3)(x+2)$

and zeros $\boxed{x = -1, 3, -2}$

2. $f(x) = x^3 - 9x^2 + 20x - 12$

$$\frac{p}{q} = \pm\frac{(1,2,3,4,6,12)}{1} = \pm(1,2,3,4,6,12)$$

$$\begin{array}{r|rrrr} 1 & 1 & -9 & 20 & -12 \\ & & 1 & -8 & 12 \\ \hline & 1 & -8 & 12 & 0 \end{array}$$

$f(x) = (x-1)(x^2 - 8x + 12)$

$= (x-1)(x-6)(x-2)$

with zeros $\boxed{x = 1, 6, 2}$

3. $c(x) = 2x^3 + 3x^2 - 1$

$$\frac{p}{q} = \pm\frac{(1)}{(1,2)} = \pm(1, \tfrac{1}{2})$$

$$\begin{array}{r|rrrr} 1 & 2 & 3 & 0 & -1 \\ & & 2 & 5 & 5 \\ \hline & 2 & 5 & 5 & 4 \end{array}$$

$$\begin{array}{r|rrrr} -1 & 2 & 3 & 0 & -1 \\ & & -2 & -1 & 1 \\ \hline & 2 & 1 & -1 & 0 \end{array}$$

$f(x) = (x+1)(2x^2 + x - 1)$

$= (x+1)(2x-1)(x+1)$

zeros $\boxed{x = -1, \tfrac{1}{2}, -1}$

Example: Find all real solutions of $x^4 - 13x^2 - 12x = 0$

factor: $x(x^3 - 13x - 12) = 0$

possible rational zeros of $x^3 - 13x - 12$ are $\frac{p}{q} = \frac{\pm(1, 2, 3, 4, \cancel{8}, 6, 12)}{1}$ not a factor

Try some to get started

$$\begin{array}{r|rrrr} 1 & 1 & 0 & -13 & -12 \\ & & 1 & 1 & -12 \\ \hline & 1 & 1 & -12 & -24 \end{array} \qquad \begin{array}{r|rrrr} -1 & 1 & 0 & -13 & -12 \\ & & -1 & 1 & 12 \\ \hline & 1 & -1 & -12 & 0 \end{array}$$

factor again:

$$x(x+1)(x^2 - x - 12) = 0$$

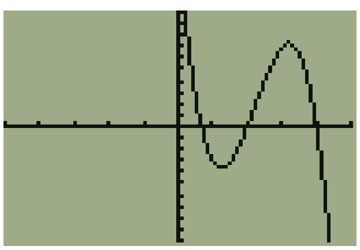
and again

$$x(x+1)(x-4)(x+3) = 0$$

From factors to zeros to get our solutions: $\boxed{x = 0, -1, 4, -3}$

Example: List the possible rational zeros of $f(x) = -3x^3 + 20x^2 - 36x + 16$. Sketch the graph of *f* so that some of the possible zeros can be disregarded and then determine all real zeros of *f*.

all possible $\frac{p}{q} = \frac{\pm(1, 2, 4, 8, 16)}{\pm(1, 3)} = \pm\left(1, 2, 4, 8, 16, \frac{1}{3}, \frac{2}{3}, \frac{4}{3}, \frac{8}{3}, \frac{16}{3}\right)$



Based on this graph, all zeros lie in the interval $[0, 5]$

$$\begin{array}{r|rrrr} 2 & -3 & 20 & -36 & 16 \\ & & -6 & 28 & -16 \\ \hline & -3 & 14 & -8 & 0 \end{array}$$

$$\begin{array}{r|rrr} 4 & -3 & 14 & -8 \\ & & -12 & 8 \\ \hline & -3 & 2 & 0 \end{array}$$

← from previous Quotient

$-3x + 2$

$f(x) = (x-2)(x-4)(-3x+2)$

Zeros: $\boxed{x = 2, 4, \frac{2}{3}}$

Complex Zeros Occur in Conjugate Pairs – Let *f(x)* be a polynomial function that has real coefficients. If $a + bi$, where $b \neq 0$, is a zero of the function, the conjugate $a - bi$ is also a zero of the function.

So if $4+5i$ is a zero, so is $4-5i$. Always in pairs.

Factors of a Polynomial – Every polynomial of degree $n > 0$ with real coefficients can be written as the product of linear and quadratic factors with real coefficients, where the quadratic factors have no real zeros.

Examples: Find a polynomial function with real coefficients that has the given zeros.

1. $4, -3i$ zeros $\rightarrow 4, -3i, +3i$

factors $\rightarrow (x-4)(x+3i)(x-3i)$

polynomial $\rightarrow f(x) = (x-4)(x+3i)(x-3i)$

2. $5, 3, -2i$

zeros $\rightarrow 5, 3, -2i, +2i$

factors $\rightarrow (x-5)(x-3)(x+2i)(x-2i)$

polynomial $\rightarrow f(x) = (x-5)(x-3)(x+2i)(x-2i)$

3. $-5, -5, 1+\sqrt{3}i$

zeros $\rightarrow -5, -5, 1+\sqrt{3}i, 1-\sqrt{3}i$

factors $\rightarrow (x+5)(x+5)(x-1+\sqrt{3}i)(x-1-\sqrt{3}i)$

polynomial $\rightarrow f(x) = (x+5)^2(x-1+\sqrt{3}i)(x-1-\sqrt{3}i)$

4. You try it: $3, -4, 5i, i$

Descartes's Rule of Signs – Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$ be a polynomial with real coefficients and $a_0 \neq 0$.

1. The number of positive real zeros of *f* is either equal to the number of variations in sign of *f(x)* or less than that number by an even integer.

2. The number of negative real zeros of *f* is either equal to the number of variations in sign of $f(-x)$ or less than that number by an even integer.

Examples: Use Descartes's Rule of Signs to determine the possible numbers of positive and negative zeros of the function.

1. $h(x) = 4x^2 - 8x + 3$

   $+ \quad - \quad +$

   1 2

   2 positive real zeros or 0 pos. real zeros

   $h(-x) = 4(-x)^2 - 8(-x) + 3$

   $= 4x^2 + 8x + 3$

   $+ \quad + \quad +$

   No sign change, no negative real zeros

2. $f(x) = -5x^3 + x^2 - x + 5$

   $- \quad + \quad - \quad +$

   3 positive real zeros or 1 real zero

   $f(-x) = 5x^3 + x^2 + x + 5$

   $+ \quad + \quad + \quad +$

   No negative real zeros

This will help us after we have all possible rational zeros to know if we should try positive or negative values. Always less two due to conjugate pairs.

Upper and Lower Bound Rules – Let *f(x)* be a polynomial with real coefficients and a positive leading coefficient. Suppose *f(x)* is divided by $x - c$, using synthetic division.

1. If *c > 0* and each number in the last row is either positive or zero, *c* is an upper bound for the real zeros of *f*.

2. If *c < 0* and the numbers in the last row are alternately positive and negative (zeros count as positive or negative), *c* is a lower bound for the real zeros of *f*.

Examples: Verify the upper and lower bounds of the real zeros of *f*.

1. $f(x) = x^3 + 3x^2 - 2x + 1$ a) Upper: $x = 1$ b) Lower: $x = -4$

$$\begin{array}{r|rrrr} 1 & 1 & 3 & -2 & 1 \\ & & 1 & 4 & 2 \\ \hline & 1 & 4 & 2 & 3 \end{array}$$

all positive so x=1 is an upper bound on zeros. (None bigger than 1).

$$\begin{array}{r|rrrr} -4 & 1 & 3 & -2 & 1 \\ & & -4 & 4 & -8 \\ \hline & 1 & -1 & 2 & -7 \end{array}$$

bottom row alternates sign so x=-4 is a lower limit.

2. $f(x) = 2x^4 - 8x + 3$ a) Upper: $x = 3$ b) Lower: $x = -4$

$$\begin{array}{r|rrrrr} 3 & 2 & 0 & 0 & -8 & 3 \\ & & 6 & 18 & 54 & 138 \\ \hline & 2 & 6 & 18 & 46 & 141 \end{array}$$

X=3 is an upper bound

$$\begin{array}{r|rrrrr} -4 & 2 & 0 & 0 & -8 & 3 \\ & & -8 & 32 & -128 & 544 \\ \hline & 2 & -8 & 32 & -136 & 547 \end{array}$$

X=-4 is a lower bound

all possible rational $\frac{p}{q} = \pm \frac{(1,3)}{(1,2)} = \pm\left(1, 3, \frac{1}{2}, \frac{3}{2}\right)$

we know 3 isn't a zero.

Descartes Rule of Signs

$f(x) = 2x^4 + 0x^3 + 0x^2 - 8x + 3$

has 2 sign changes so 2 positive zeros

$f(-x) = 2x^4 + 8x + 3$

has no negative zeros