3.5 Exponential and Logarithmic Models

The five most common types of mathematical models involving exponential functions and logarithmic functions are as follows:

1. Exponential growth model $y = ae^{bx},\ b > 0$
2. Exponential decay model $y = ae^{-bx},\ b > 0$
3. Gaussian model $y = ae^{-(x-b)^2/c}$
4. Logistic growth model $y = \dfrac{a}{1+be^{-rx}}$
5. Logarithmic models $y = a + b\ln x, \quad y = a + b\log x$

Examples:

1. Determine the principal P that must be invested at 5%, compounded monthly, so that $500,000 will be available for retirement in 10 years.

$A = P(1+\frac{r}{n})^{nt}$

$500{,}000 = P(1+\frac{.05}{12})^{12 \times 10}$

$500{,}000 = P(1.0041\bar{6})^{120}$

$\dfrac{500{,}000}{(1.0041\bar{6})^{120}} = P = \$303580.52$

2. Determine the time necessary for $1000 to double if it is invested at 6.5% if it is compounded

$A = 2000$

a) annually

$\dfrac{2000}{1000} = \dfrac{1000(1+\frac{.065}{1})^{1 \ast t}}{1000}$

$2 = 1.065^t$

$t = \log_{1.065}(2)$

$t \approx 11$ years

b) monthly

$2000 = 1000(1+\frac{.065}{12})^{12t}$

$2 = (1.00541\bar{6})^{12t}$

$12t = \log_{1.005416}(2)$

$t = \dfrac{\log_{1.005416}(2)}{12} \approx 10.69$ yrs

c) daily

$2000 = 1000(1+\frac{.065}{365})^{365t}$

$2 = (1+\frac{.065}{365})^{365t}$

$365t = \log_{(1+\frac{.065}{365})}(2)$

$t = \dfrac{\log_{(1+\frac{.065}{365})}(2)}{365} \approx 10.66$ yrs

3. Carbon 14 decays with a half-life of 5715 years. Find how much remains from a 6.5 g sample after 1000 years.

exponential decay: $y = ae^{-bx}$ where a is initial amount, b is decay constant and x is time

half-life: amount of time for ½ substance to decay.

Strategy: we know $a = 6.5g$ but we must find decay constant b. Then we use $a + b$ to find y.

$$\frac{\frac{1}{2}a}{a} = \frac{ae^{-b(5715)}}{a} \longrightarrow \frac{1}{2} = e^{-5715b} \longrightarrow -5715b = \ln(\tfrac{1}{2}) \rightarrow b = \frac{\ln(\frac{1}{2})}{-5715} = 1.213 \times 10^{-4}$$

so $b = 0.0001213$

Now $y = 6.5e^{-0.0001213(1000)} = 5.7576$ or approx 5.8 grams

4. The populations P (in thousands) of Horry County, South Carolina from 1970 through 2007 can be modeled by $P = -18.5 + 92.2e^{0.0282t}$, where *t* represents the year, with *t = 0* corresponding to 1970.

a) Find the populations for years 1970, 1980, 1990, 2000, and 2007.

Years since 1970	0	10	20	30	37
Population in thousands	73.7 *	103.7	143.6	196.4	243.2

*Not 73.7 people, but 73.7 thousand people = $73.7 \times 1000 = 73{,}700$

b) According to the model, when will the population of Horry County reach 300,000?

300,000 = 300 thousand so we solve $300 = -18.5 + 92.2e^{0.0282t}$ for t.

$$\begin{aligned} 300 &= -18.5 + 92.2e^{0.0282t} \\ +18.5 & \quad +18.5 \\ \frac{318.5}{92.2} &= \frac{92.2e^{0.0282t}}{92.2} \end{aligned}$$

$$\frac{318.5}{92.2} = e^{0.0282t} \longrightarrow 0.0282t = \ln\left(\frac{318.5}{92.2}\right) \longrightarrow t = \frac{\ln\left(\frac{318.5}{92.2}\right)}{0.0282}$$

$t \approx 43.9596576$ or 44 yrs from 1970 → In the year 2014.

c) Do you think the model is valid for long-term predictions of the population?

What do you think and why?

5. The number of bacteria in a culture is increasing according to the law of exponential growth. After 3 hours, there are 100 bacteria, and after 5 hours, there are 400 bacteria. How many bacteria will there be after 6 hours?

$y = ae^{bx}$

We do not know initial amount or how quickly they grow, growth rate. But we do know two values:

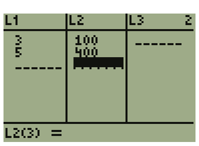
3 hrs, 100 bacteria gives $100 = ae^{b(3)}$ or $100 = ae^{3b}$

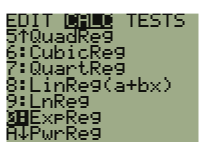
5 hrs, 400 bacteria gives $400 = ae^{5b}$. The a + b in each equation is the same so we solve one equation for a: $\frac{100}{e^{3b}} = a$ then substitute it into the other equation: $400 = \frac{100}{e^{3b}} \cdot e^{5b}$. Now we have one equation with one variable and we can find the value of b.

$$400 = \frac{100e^{5b}}{e^{3b}} \rightarrow \frac{400}{100} = \frac{100e^{2b}}{100} \rightarrow 4 = e^{2b} \rightarrow 2b = \ln 4 \text{ so } b = \frac{\ln(4)}{2} \approx 0.6931$$

With b, we can find a in the blue equation: $\frac{100}{e^{3(.6931)}} = a \approx 12.5$

But we're not finished yet! The question asks how many bacteria after 6 hours. $y = ae^{bx}$ at 6 hours is $y = 12.5e^{0.6931(6)} \approx 799.8$ or 800 bacteria





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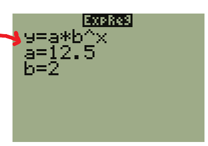

You can also use a graphing calculator to find an exponential regression. Be careful! My calculator gave a = 12.5 and b = 2. You must pay attention to the form of the answer.*

We used $y = ae^{bx}$ to find a and b. The calculator uses $y = a(b)^x$.

Either way, $y = 12.5(2)^6 = 800$.

6. At 8:30 A.M., a coroner was called to the home of a person who had died during the night. In order to estimate the time of death, the coroner took the person's temperature twice. At 9:00 A.M. the temperature was 85.7 degrees, and at 11:00 A.M. the temperature was 82.8 degrees. From these two temperatures, the coroner was able to determine that the time elapsed since death and the body temperature were related by the formula $t = -10\ln\frac{T-70}{98.6-70}$ where t is the time in hours elapsed since the person died and T is the temperature (in degrees F) of the person's body. (This formula is derived from Newton's Law of Cooling.) Use the formula to estimate the time of death of the person.

$$t = -10\ln\frac{85.7-70}{98.6-70} = -10\ln\frac{15.7}{28.6} = 5.99746 \approx 6 \text{ hours earlier}$$

$$\begin{array}{r} 9{:}00\text{ AM} \\ -6\ \text{hrs} \\ \hline 3{:}00\text{ Am} \end{array}$$

$$t = -10\ln\frac{82.8-70}{98.6\ 70} = -10\ln\frac{12.8}{28.6} \approx 8.0396$$

$$\begin{array}{r} 11\text{ Am} \\ -8\ \text{hrs} \\ \hline 3{:}00\text{ Am} \end{array}$$

either way, time of death was approximately 3:00 Am