Chapter Four: Trigonometry

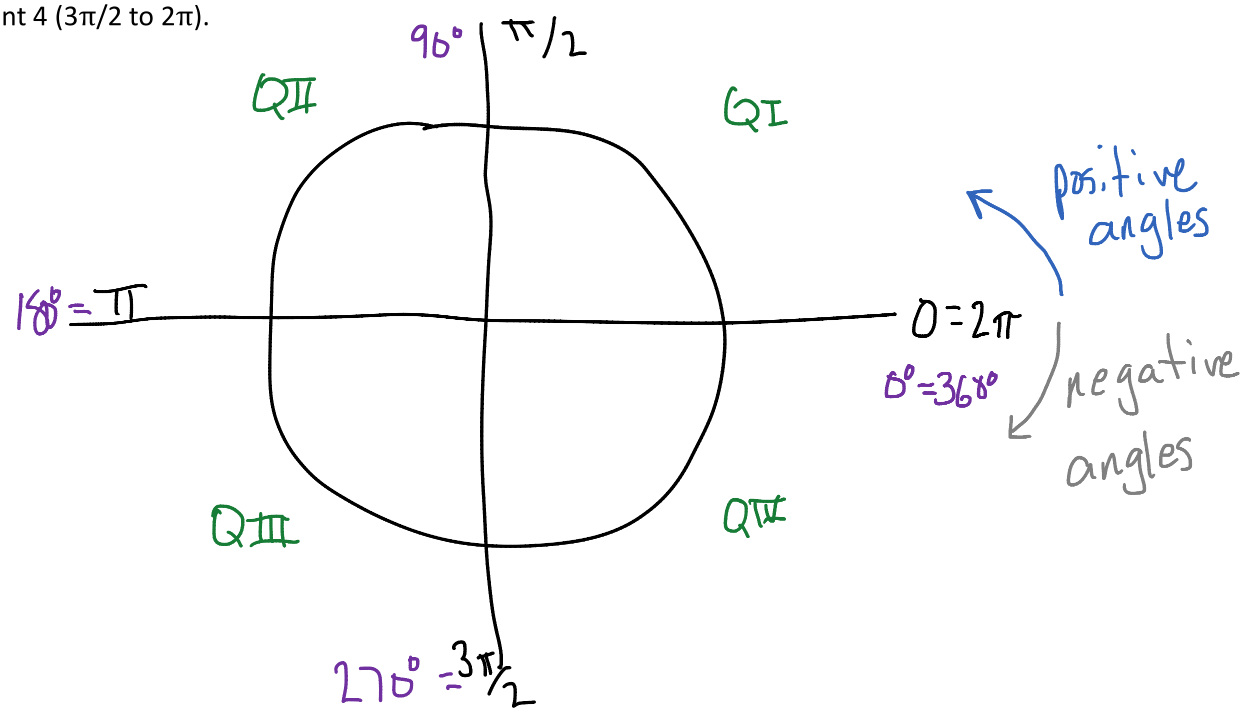
4.1 Radian and Degree Measure

What is trigonometry? From the Greek it literally means "measuring triangles." Before we get to triangles though, we first deal with angles.

Definitions: An angle is determined by rotating a ray (half-line) about its endpoint. The starting position of the ray is the initial side of the angle, and the position after rotation is the terminal side. The endpoint of the ray is the vertex of the angle. If the vertex is at the origin and the initial side of the angle is on the positive x-axis, the angle is said to be in standard position. Positive angles are made by a counterclockwise rotation and negative angles are generated by a negative rotation. For any angle we can measure the positive and negative rotations therefore finding angles that are coterminal. Generally, angles are labeled with capital letters or Greek letters.

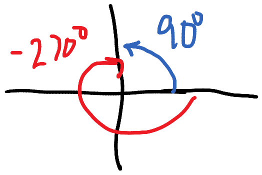
Definition: The measure of an angle is determined by the amount of rotation from the initial side to the terminal side. One radian is the measure of a central angle θ that intercepts an arc *s* equal in length to the radius *r* of the circle. Algebraically this means that $\theta = \frac{s}{r}$ where θ is measured in radians.

Because the circumference of a circle is 2πr units, it follows that a central angle of one full revolution (counterclockwise) corresponds to an arc length of $s=2\pi r$. This means we can divide up a rectangular coordinate system into quadrant 1 (0 to π/2), quadrant 2 (π/2 to π), quadrant 3 (π to 3π/2) and finally quadrant 4 (3π/2 to 2π).



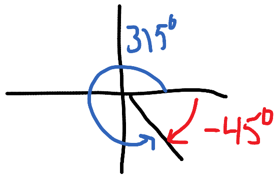
Examples: Find angles that are coterminal.

1. 90°



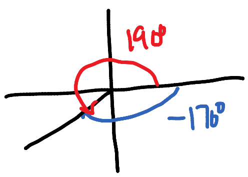
$90^\circ - 360^\circ = -270^\circ$

2. 315°



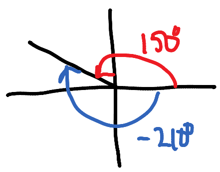
$315^\circ - 360^\circ = -45^\circ$

3. -170°



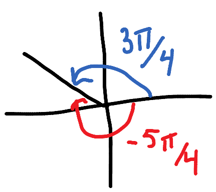
$360^\circ - 170^\circ = 190^\circ$

4. -210°



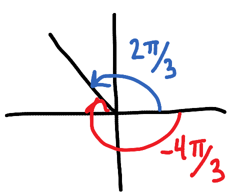
$360^\circ - 210^\circ = 150^\circ$

5. $\frac{3\pi}{4}$



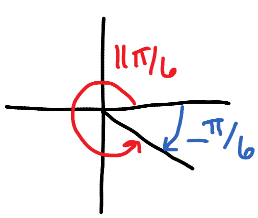
$\frac{3\pi}{4} - 2\pi = \frac{3\pi}{4} - \frac{8\pi}{4} = \frac{-5\pi}{4}$

6. $\frac{2\pi}{3}$



$\frac{2\pi}{3} - 2\pi = \frac{2\pi}{3} - \frac{6\pi}{3} = -\frac{4\pi}{3}$

7. $-\frac{\pi}{6}$



$2\pi - \frac{\pi}{6} = \frac{12\pi}{6} - \frac{\pi}{6} = \frac{11\pi}{6}$

Facts: Two positive angles are complementary if their sum is $\pi/2$. Two positive angles are supplementary if their sum is π.

Examples: Find the complements and supplements, if possible.

Given Angle	Complement	Supplement
30°	$90^\circ - 30^\circ = 60^\circ$	$180^\circ - 30^\circ = 150^\circ$
50°	$90^\circ - 50^\circ = 40^\circ$	$180^\circ - 50^\circ = 130^\circ$
110°	$90^\circ - 110^\circ = \cancel{-20^\circ}$ none	$180^\circ - 110^\circ = 70^\circ$
$\frac{\pi}{4}$	$\frac{\pi}{2} - \frac{\pi}{4} = \pi/4$	$\pi - \frac{\pi}{4} = 3\pi/4$
$\frac{2\pi}{3}$	$\frac{\pi}{2} - \frac{2\pi}{3} =$ negative none	$\pi - \frac{2\pi}{3} = \pi/3$
$\frac{\pi}{6}$	$\frac{\pi}{2} - \frac{\pi}{6} = \frac{2\pi}{6} = \frac{\pi}{3}$	$\pi - \frac{\pi}{6} = \frac{5\pi}{6}$

Definition: A second way to measure angles is through degrees. Since a whole circle corresponds to 360°, a single degree is 1/360 of a circle. Also, 360° [illegible] rad.

$360^\circ = 2\pi$ rad

Conversions between Degrees and Radians:

1. To convert degrees to radians, multiply degrees by $\frac{\pi \, rad}{180^\circ}$.

2. To convert radians to degrees, multiply radians by $\frac{180^\circ}{\pi \, rad}$.

Examples: Convert as needed.

1. 315°

$315^\circ \cdot \frac{\pi}{180^\circ} = \frac{315\pi}{180} = \frac{7\pi}{4}$

2. 240°

$240^\circ \cdot \frac{\pi}{180^\circ} = \frac{240\pi}{180} = \frac{4\pi}{3}$

3. $\frac{\pi}{3}$

$\frac{\pi}{3} \cdot \frac{180}{\pi} = \frac{180\pi}{3\pi} = 60^\circ$

4. $\frac{7\pi}{6}$

$\frac{7\pi}{\cancel{6}_{1}} \cdot \frac{\cancel{180}^{30}}{\pi} = 210^\circ$

Just like time, degrees have subdivisions of minutes and seconds. There are 60 minutes = 60’ in one degree and there are 60 seconds = 60” in one minute. Sometimes we use degrees, minutes, seconds (DMS) and sometimes we use decimal degrees. It is important to know how to convert these in both directions.

Examples: Convert as needed.

1. 46.25°

$46^\circ + .25^\circ$ $.25$ of 60 is

$.25(60) = 15$

$\underline{46^\circ 15'}$

2. 109.15°

$.15^\circ = .15(60)$ minutes

$= 9$

$109^\circ 9'$

3. $72^\circ 39'$

$39' = \frac{39}{60}$ degrees

$= 0.65$

72.65°

4. $-114^\circ 19'$

$19' = \frac{19}{60}$ degrees

$= 0.31666\ldots$

-114.32° (to two decimal places)

Definition: For a circle of radius *r*, a central angle θ intercepts an arc length of *s* given by $s = r\theta$ where θ is measured in radians. Note: You must use radians!!!

Examples: Find the arc length.

1. A circle has a radius of 3 inches. Find the length of the arc intercepted by a central angle of 150°.

1) Convert $150^\circ \cdot \frac{\pi}{180^\circ} = \frac{150\pi}{180} = \frac{15\pi}{18} = \frac{5\pi}{6}$

2) Use $s = r\theta = 3\left(\frac{5\pi}{6}\right) = \frac{15\pi}{6} = \boxed{\frac{5\pi}{2}}$

This is your answer. No decimals in trig unless it tells you to round.

2. A circle has a radius of 4 inches. Find the length of the arc intercepted by a central angle of 45°.

1) Convert $45^\circ \cdot \frac{\pi}{180^\circ} = \frac{\pi}{4}$

2) $s = r\theta = 4\left(\frac{\pi}{4}\right) = \pi$

Examples: Find the central angle.

1. A circle has a radius of 20 miles. Find the radian measure of the central angle intercepted by an arc length of 100 miles.

Use $s = r\theta$

$100 = 20\theta$

$5 \text{ rad} = \theta$

2. A circle has a radius of 125 km. Find the radian measure of the central angle intercepted by an arc length of 1000 km.

$s = r\theta$

$1000 = 125\theta$

$\frac{1000}{125} = \theta = 8 \text{ rad}$

Linear and Angular Speeds: Consider a particle moving at a constant speed along a circular arc of radius *r*. If *s* is the length of the arc traveled in time *t*, then the linear speed *v* of the particle is

$$Linear\ speed\ v = \frac{arc\,length}{time} = \frac{s}{t}$$

Moreover, if θ is the angle (in radian measure) corresponding to the arc length *s*, then the angular speed ω of the particle is

$$Angular\ speed\ \omega = \frac{central\ angle}{time} = \frac{\theta}{t}$$

Example: A carousel with a 50 foot diameter makes 4 revolutions per minute.

1. Find the angular speed of the carousel in radians per minute.

4 revolutions per minute is $4(2\pi) = 8\pi$ radians per minute

2. Find the linear speed (in feet per minute) of the platform rim of the carousel.

$25(4)(2\pi) = 200\pi$ ft/min

↓

diameter = 50 so radius = 25

Area of a Sector of a Circle: For a circle of radius *r*, the area A of a sector of the circle with central angle θ is given by $A = \frac{1}{2}r^2\theta$ where θ is measured in radians.

Examples: Find the area of the sector of the circle with radius *r* and central angle θ.

1. r = 6 inches, θ = π/3

$$A = \frac{1}{2}(6)^2\left(\frac{\pi}{3}\right) = \frac{1}{2}(36)\frac{\pi}{3} = 18\frac{\pi}{3} = 6\pi \text{ in}^2$$

2. r = 2.5 feet, θ = 225°

$$A = \frac{1}{2}(2.5)^2\left(\frac{5\pi}{4}\right) = \frac{1}{2}(6.25)\frac{5\pi}{4} = \frac{1}{2}\left(\frac{25}{4}\right)\left(\frac{5\pi}{4}\right) = \frac{125\pi}{32}$$

Convert first

$$225^\circ = 225 \cdot \frac{\pi}{180} = \frac{5\pi}{4}$$