4.4 Trigonometric Functions of Any Angle

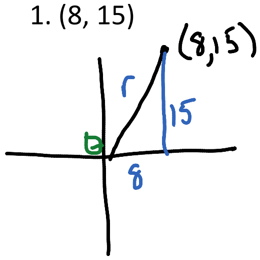
Definition of Trig Functions of Any Angle – Let θ be an angle in standard position with (x, y) a point on the terminal side of θ and $r = \sqrt{x^2 + y^2} \neq 0$.

$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r} \qquad \tan\theta = \frac{y}{x},\ x \neq 0$$

$$\csc\theta = \frac{r}{y},\ y \neq 0 \quad \sec\theta = \frac{r}{x},\ x \neq 0 \quad \cot\theta = \frac{x}{y},\ y \neq 0$$

Examples: The point is on the terminal side of an angle in standard position. Determine the exact values of the six trigonometric functions of the angle.

1. (8, 15)



$r^2 = 8^2 + 15^2$

$r^2 = 64 + 225$

$r^2 = 289$

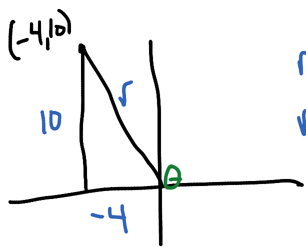
$r = 17$

$\sin\theta = \frac{15}{17}$ $\qquad \csc\theta = \frac{17}{15}$

$\cos\theta = \frac{8}{17}$ $\qquad \sec\theta = \frac{17}{8}$

$\tan\theta = \frac{15}{8}$ $\qquad \cot\theta = \frac{8}{15}$

2. (-4, 10)



$r^2 = (-4)^2 + 10^2$

$r^2 = 16 + 100$

$r^2 = 116$

$r = \sqrt{116} = 2\sqrt{29}$

$\sin\theta = \frac{10}{2\sqrt{29}} = \frac{5}{\sqrt{29}} = \frac{5\sqrt{29}}{29}$ $\qquad \csc\theta = \frac{\sqrt{29}}{5}$

$\cos\theta = \frac{-4}{2\sqrt{29}} = -\frac{2}{\sqrt{29}} = \frac{-2\sqrt{29}}{29}$ $\qquad \sec\theta = \frac{\sqrt{29}}{-2}$

$\tan\theta = \frac{10}{-4} = -\frac{5}{2}$ $\qquad \cot\theta = -\frac{2}{5}$

Examples: State the quadrant in which θ lies.

1. $\sin\theta > 0$ *and* $\cos\theta > 0$

$\downarrow$ sin θ is y

y > 0 in QI + II

$\downarrow$ cos θ is x

x > 0 in QI + IV

QI

2. $\sin\theta > 0$ *and* $\cos\theta < 0$

y > 0

I + II

x < 0

II + III

QII

Example: Find the values of the six trig functions of θ with the given constraint.

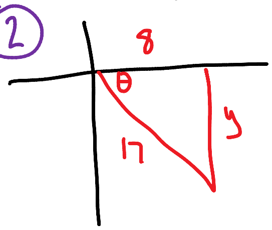
1. $\cos\theta = \dfrac{8}{17}$ *with* $\tan\theta < 0$

(1) $\tan\theta < 0$ when x & y have opposite signs QII + IV

$\cos\theta$ is positive when $x > 0$ so QI + IV

use QIV

(2)



$8^2 + y^2 = 17^2$

$64 + y^2 = 289$

$y^2 = 225$

$y = -15$ (why -?)

(3) $\sin\theta = \dfrac{-15}{17}$ $\csc\theta = \dfrac{-17}{15}$

$\cos\theta = \dfrac{8}{17}$ $\sec\theta = \dfrac{17}{8}$

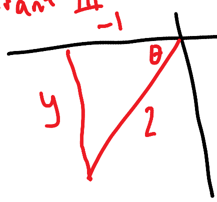
$\tan\theta = \dfrac{-15}{8}$ $\cot\theta = -\dfrac{8}{15}$

2. $\sec\theta = -2$ *with* $\sin\theta < 0$

$\sin\theta < 0$ in quadrants III & IV

$\sec\theta$ + $\cos\theta$ are negative in II + III

use quadrant III



$2^2 = (-1)^2 + y^2$

$4 = 1 + y^2$

$3 = y^2$

$-\sqrt{3} = y$

$\sin\theta = \dfrac{-\sqrt{3}}{2}$ $\csc\theta = -\dfrac{2}{\sqrt{3}} = \dfrac{-2\sqrt{3}}{3}$

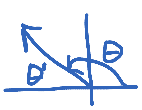
$\cos\theta = -\dfrac{1}{2}$ $\sec\theta = -2$

$\tan\theta = \dfrac{-\sqrt{3}}{-1} = \sqrt{3}$ $\cot\theta = \dfrac{-1}{-\sqrt{3}} = \dfrac{\sqrt{3}}{3}$

Definition of Reference Angle – Let θ be an angle in standard position. Its reference angle is the acute angle θ’ formed by the terminal side of θ and the horizontal axis. aka: the x-axis

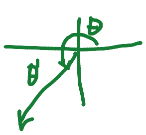
QI – the reference angle $\theta' = \theta$.

QII



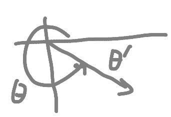
the two angles are supplementary: $\theta + \theta' = \pi$

QIII



the reference angle is the little bit of θ that went past π: $\theta - \pi = \theta'$

QIV

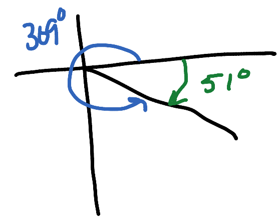


the two angles together make 1 full rotation: $\theta + \theta' = 2\pi$

you can use these formulas to solve for either θ or θ' as needed.

Examples: Find the reference angle and sketch both angles.

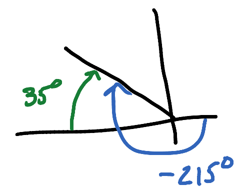
1. $\theta = 309^\circ$



309° is bigger than 270°, less than 360° so in Q IV

In QIV $309 + \theta' = 360$ so $\theta' = 360 - 309 = 51^\circ$

2. -215°

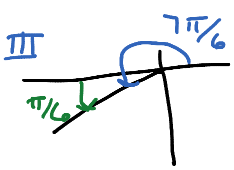


-215 is more than -180, less than -270 so in Q II

a coterminal angle is $360^\circ - 215^\circ = 145^\circ$

In QII $145^\circ + \theta' = 180^\circ$ so $\theta' = 180 - 145 = 35^\circ$

3. $\theta = \dfrac{7\pi}{6}$



$\frac{7\pi}{6}$ is $\frac{7}{6}$ of π. $\frac{7}{6}$ is more than 1, less than $\frac{3}{2}$ so Q III

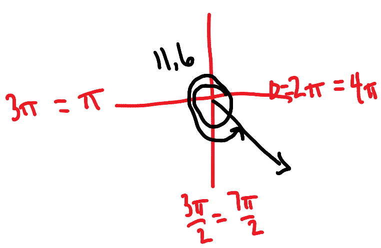
In Q3: $\frac{7\pi}{6} - \pi = \frac{7\pi}{6} - \frac{6\pi}{6} = \frac{\pi}{6}$

4. $\theta = 11.6$

no degree symbol so this is in radians

$\pi \approx 3.14$, $2\pi \approx 6.28$, $3\pi \approx 9.42$ $4\pi \approx 12.56$ → somewhere between $3\pi + 4\pi$

$3.5\pi \approx 10.99$



Now that we know the quadrant, let's find a coterminal angle in the interval $[0, 2\pi)$: $11.6 - 2\pi = 5.32$

In QIV: $2\pi - 5.32 = 0.96$ so $\theta' = 0.96$ rad

Evaluating Trigonometric Functions of Any Angle – To find the value of a trigonometric function of any angle θ:

1. Determine the function value for the associated reference angle θ'.

2. Depending on the quadrant in which θ lies, affix the appropriate sign to the function value.

Know signs (+ or -) and reference angles so you only need to remember 1st quadrant of unit circle

Examples: Find two solutions of the equation. Give your answers in degrees and radians.

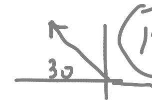
1. $\sin\theta = \frac{1}{2}$

Sin θ is positive in QI + II

$\sin\theta = \frac{1}{2}$ when $\theta = 30°$ ← reference

reference → $30°$ in quadrant I is $30°$

$30°$ in QII is $150°$



in radians $\boxed{\begin{matrix} \frac{\pi}{6} \\ \frac{5\pi}{6} \end{matrix}}$

2. $\csc\theta = \frac{2\sqrt{3}}{3}$

$\csc\theta = \frac{2}{\sqrt{3}}$ so $\sin\theta = \frac{\sqrt{3}}{2}$

positive QI + II

$\sqrt{3}/2$ at $\pi/3 = 60°$

QI : $\frac{\pi}{3} = 60°$

QII : $180 - 60 = 120°$

$\pi - \frac{\pi}{3} = \frac{2\pi}{3}$

3. $\tan\theta = 1$

positive QI + III

1 when sin θ + cos θ are equal at $\frac{\pi}{4} = 45°$

QI : $\frac{\pi}{4} = 45°$

QIII : $\pi + \frac{\pi}{4} = \frac{5\pi}{4}$

$180 + 45 = 225°$