5.5 Multiple-Angle and Product-to-Sum Formulas

Double-Angle Formulas

$$\sin 2u = 2\sin u \cos u \qquad\qquad \tan 2u = \frac{2\tan u}{1-\tan^2 u}$$

$$\begin{aligned}\cos 2u &= \cos^2 u - \sin^2 u \\ &= 2\cos^2 u - 1 \\ &= 1 - 2\sin^2 u\end{aligned}$$

Example: Solve $2\cos x + \sin 2x = 0$

$$2\cos x + \sin 2x = 0$$
$$2\cos x + 2\sin x\cos x = 0$$
$$2\cos x(1+\sin x) = 0$$

$$2\cos x = 0 \quad \text{or} \quad 1 + \sin x = 0$$
$$\cos x = 0 \qquad\qquad \sin x = -1$$
$$x = \frac{\pi}{2}, \frac{3\pi}{2} \qquad\qquad x = \frac{3\pi}{2}$$

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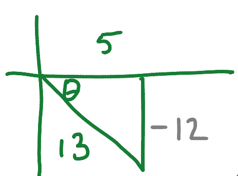
Example: If $\cos\theta = \dfrac{5}{13}$, with theta in the fourth quadrant, find $\sin 2\theta, \cos 2\theta,$ and $\tan 2\theta$.



$$\sin 2\theta = 2\sin\theta\cos\theta = 2\left(\frac{-12}{13}\right)\left(\frac{5}{13}\right) = 2\left(\frac{-60}{169}\right) = \frac{-120}{169}$$

choose one, use given info when possible →

$$\cos 2\theta = 2\cos^2\theta - 1 = 2\left(\frac{5}{13}\right)^2 - 1 = 2\left(\frac{25}{169}\right) - 1$$
$$= \frac{50}{169} - \frac{169}{169} = \frac{-119}{169}$$

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} = \frac{2\left(\frac{-12}{5}\right)}{1-\left(\frac{-12}{5}\right)^2} = \frac{\frac{-24}{5}}{1-\left(\frac{144}{25}\right)}$$
$$= \frac{\frac{-24}{5}}{\frac{25-144}{25}} = \frac{-24}{5}\cdot\frac{25}{-119} = \frac{120}{119}$$

Power-Reducing Formulas

$$\sin^2 u = \frac{1-\cos 2u}{2} \qquad \cos^2 u = \frac{1+\cos 2u}{2} \qquad \tan^2 u = \frac{1-\cos 2u}{1+\cos 2u}$$

Half-Angle Formulas

$$\sin\frac{u}{2} = \pm\sqrt{\frac{1-\cos u}{2}} \qquad \cos\frac{u}{2} = \pm\sqrt{\frac{1+\cos u}{2}} \qquad \tan\frac{u}{2} = \frac{1-\cos u}{\sin u} = \frac{\sin u}{1+\cos u}$$

The signs of the sine and cosine depend on the quadrant that u/2 lies in.

Example: Find the exact value of $\sin 105^\circ$.

$$\sin 105^\circ = \sin\frac{210^\circ}{2} = \pm\sqrt{\frac{1-\cos 210^\circ}{2}} = +\sqrt{\frac{1-\left(-\frac{\sqrt{3}}{2}\right)}{2}} = \sqrt{\frac{\frac{2+\sqrt{3}}{2}}{2}}$$

use + because 105° is in Q2

$$\frac{\sqrt{2+\sqrt{3}}}{2} = \sqrt{\frac{2+\sqrt{3}}{4}}$$

Example: Find all solutions of $2-\sin^2 x = 2\cos^2\frac{x}{2}$ in the interval $[0, 2\pi)$.

$$\cos\frac{x}{2} = \pm\sqrt{\frac{1+\cos x}{2}} \quad \text{so} \quad \cos^2\frac{x}{2} = \frac{1+\cos x}{2} \qquad \sin^2 x = 1-\cos^2 x$$

$$2-\sin^2 x = 2\cos^2\frac{x}{2} \quad \text{becomes} \quad 2-(1-\cos^2 x) = 2\left(\frac{1+\cos x}{2}\right)$$

$$1+\cos^2 x = 1+\cos x$$

$$\cos^2 x - \cos x = 0$$

$$\cos x(\cos x - 1) = 0$$

$$\cos x = 0 \quad \text{or} \quad \cos x = 1$$

$$x = \frac{\pi}{2}, \frac{3\pi}{2} \qquad x = 0$$

$x = 2\pi$ is out of the given interval

Product-to-Sum Formulas

$$\sin u \sin v = \frac{1}{2}\left[\cos(u-v) - \cos(u+v)\right] \qquad \cos u \cos v = \frac{1}{2}\left[\cos(u-v) + \cos(u+v)\right]$$

$$\sin u \cos v = \frac{1}{2}\left[\sin(u+v) + \sin(u-v)\right] \qquad \cos u \sin v = \frac{1}{2}\left[\sin(u+v) - \sin(u-v)\right]$$

Sum-to-Product Formulas

$$\sin u + \sin v = 2\sin\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right) \qquad \sin u - \sin v = 2\cos\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right)$$

$$\cos u + \cos v = 2\cos\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right) \qquad \cos u - \cos v = -2\sin\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right)$$

Example: Find the exact value of $\cos 195^\circ + \cos 105^\circ$.

$$\begin{aligned}
\cos 195^\circ + \cos 105^\circ &= 2\cos\left(\frac{195^\circ + 105^\circ}{2}\right)\cos\left(\frac{195 - 105^\circ}{2}\right) \\
&= 2\cos(150^\circ)\cos(45^\circ) \\
&= 2\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) \\
&= \sqrt{3}\left(\frac{\sqrt{2}}{2}\right) \\
&= \frac{\sqrt{6}}{2}
\end{aligned}$$