Chapter Six: Additional Topics in Trigonometry

6.1 Law of Sines

All the trig rules we've learned so far have been for right triangles. But there are also acute and obtuse triangles. How can we solve these?

To solve an oblique triangle, you need to know the measure of at least one side and any two other measures of the triangle – two sides, two angles, or one angle and one side. The breaks down into the following four cases:

1. Two angles and any side (AAS or ASA).
2. Two sides and an angle opposite one of them (SSA).
3. Three sides (SSS).
4. Two sides and their included angle (SAS).

The first two cases can be solved using the Law of Sines, whereas the last two cases require the Law of Cosines.

Law of Sines – If *ABC* is a triangle with sides *a, b,* and *c,* then

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

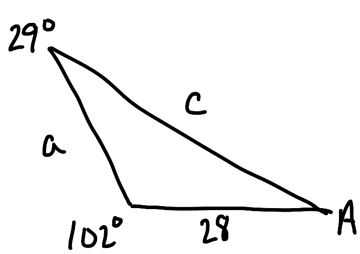
We can also write this in its reciprocal form if it is better for the given problem.

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

use whichever form you need. the missing value is easier in the numerator

Examples: Solve the given triangles.

1. If $C = 102^\circ$, $B = 29^\circ$ and $b = 28$ ft., find the remaining angle and sides.

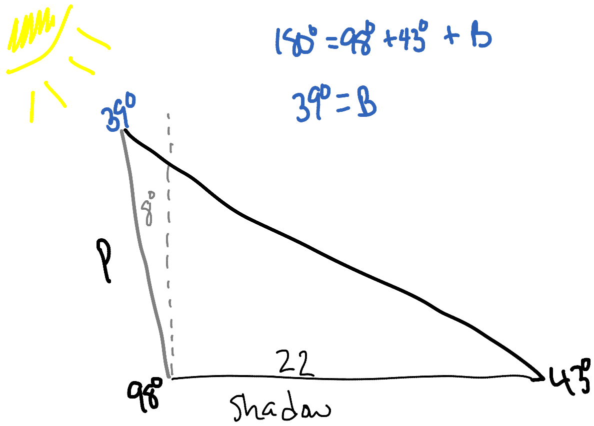


$A + 29^\circ + 102^\circ = 180^\circ$ so $A = 49^\circ$

$$\frac{c}{\sin 102^\circ} = \frac{28}{\sin 29^\circ} \Rightarrow c = \frac{28 \sin 102^\circ}{\sin 29^\circ} = 56.5 \text{ ft}$$

$$\frac{a}{\sin 49^\circ} = \frac{28}{\sin 29^\circ} \Rightarrow a = \frac{28 \sin 49^\circ}{\sin 29^\circ} = 43.6 \text{ ft}$$

2. A pole tilts toward the sun at an 8° angle from the vertical, and it casts a 22-foot shadow. The angle of elevation from the tip of the shadow to the top of the pole is 43°. How tall is the pole?



$180^\circ = 98^\circ + 43^\circ + B$

$39^\circ = B$

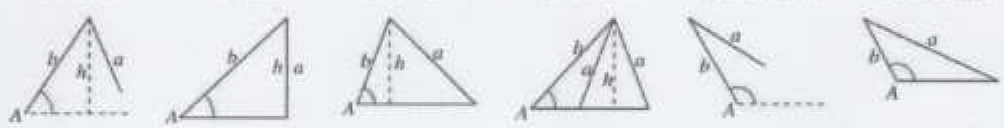
$$\frac{p}{\sin 43^\circ} = \frac{22}{\sin 39^\circ}$$

$$p = \frac{22 \sin 43^\circ}{\sin 39^\circ} = 23.8 \text{ ft}$$

If two sides and one opposite angle are given, there are three possible situations that can occur: (1) no such triangle exists, (2) one such triangle exists, or (3) two distinct triangles may satisfy the conditions.

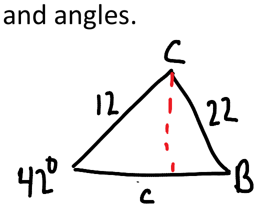
The Ambiguous Case (SSA)

Consider a triangle in which you are given a, b, and A. $(h = b \sin A)$



	A is acute.	A is acute.	A is acute.	A is acute.	A is obtuse.	A is obtuse.
Sketch						
Necessary condition	$a < h$	$a = h$	$a \geq b$	$h < a < b$	$a \leq b$	$a > b$
Triangles possible	None	One	One	Two	None	One

Single Solution (SSA) – For the triangle with $a = 22\,in, b = 12\,in,$ and $A = 42^\circ$, find the remaining side and angles.



check $h = b \sin A = 12 \sin 42^\circ = 8.03$

$a \geq b$ so one triangle

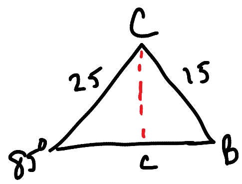
$$\frac{\sin B}{12} = \frac{\sin 42^\circ}{22} \Rightarrow \sin B = \frac{12 \sin 42^\circ}{22} \Rightarrow B = \sin^{-1}\left(\frac{12 \sin 42^\circ}{22}\right)$$

$B \approx 21.4^\circ$

$42^\circ + 21.4^\circ + C = 180^\circ$ so $C = 116.6^\circ$

$$\frac{c}{\sin 116.6^\circ} = \frac{22}{\sin 42^\circ} \Rightarrow c = \frac{22 \sin 116.6^\circ}{\sin 42^\circ} = 29.4$$

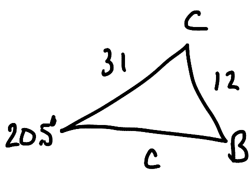
No Solution (SSA) – Show that there is no triangle for which $a = 15, b = 25, and\ A = 85^{\circ}$.



Check $h = b\sin A = 25\sin 85^{\circ} = 24.9$

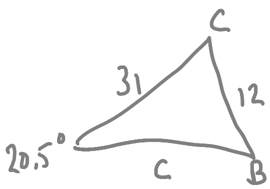
$a < h$ so no triangle possible

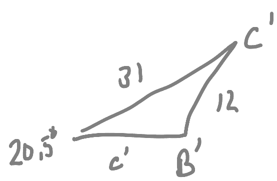
Two Solutions (SSA) – Find two triangles for which $a = 12\,m, b = 31\,m,\ and\ A = 20.5^{\circ}$.



Check $h = b\sin A = 31\sin 20.5^{\circ} = 10.9$

$h < a < b$ so two possible quadrants 1+2





$\frac{\sin B}{31} = \frac{\sin 20.5^{\circ}}{12} \Rightarrow \sin B = \frac{31\sin 20.5^{\circ}}{12} \Rightarrow \sin B = 0.904702\ldots$

$B = \sin^{-1}(0.904702\ldots) = 64.8^{\circ}$

$B' = 180^{\circ} - 64.8^{\circ} = 115.2^{\circ}$

$20.5^{\circ} + 64.8^{\circ} + C = 180$ so $C = 94.7^{\circ}$

$\frac{c}{\sin 94.7^{\circ}} = \frac{12}{\sin 20.5} \Rightarrow c = \frac{12\sin 94.7^{\circ}}{\sin 20.5^{\circ}} \Rightarrow c = 34.2$

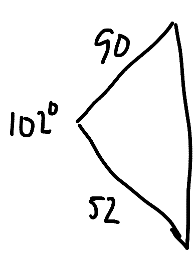
$20.5^{\circ} + 115.2^{\circ} + C' = 180$ so $C' = 44.3^{\circ}$

$\frac{c'}{\sin 44.3^{\circ}} = \frac{12}{\sin 20.5^{\circ}} \Rightarrow c' = \frac{12\sin 44.3^{\circ}}{\sin 20.5^{\circ}} \Rightarrow c' = 23.9$

Area of an Oblique Triangle – The area of any triangle is one-half the product of the lengths of two sides times the sine of their included angle. That is,

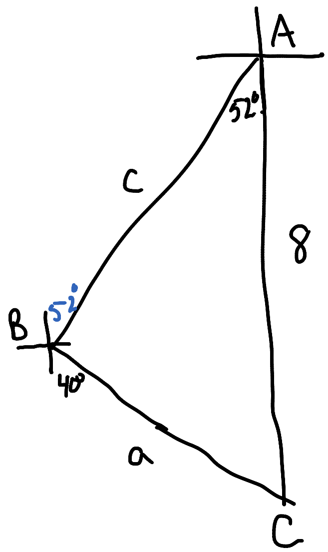
$$Area = \frac{1}{2}bc\sin A = \frac{1}{2}ac\sin B = \frac{1}{2}ab\sin C$$

Example: Find the area of a triangular lot having two sides of lengths 90 meters and 52 meters and an included angle of 102°.



$$\begin{aligned} Area &= \tfrac{1}{2}(90)(52)\sin 102^{\circ} \\ &= 45(52)\sin 102^{\circ} \\ &= 2340\sin 102^{\circ} \\ &= 2288.9\ m^2 \end{aligned}$$

Example: The course for a boat race starts at point A and proceeds in the direction S 52 degrees W to point B, then in the direction S 40 degrees E to point C, and finally back to A. Point C lies 8 kilometers directly south of point A. Approximate the total distance of the race course.



angle B is $180 - 52° - 40° = 88°$

$$\frac{a}{\sin 52°} = \frac{8}{\sin 88°} \Rightarrow a = \frac{8\sin 52°}{\sin 88°} = 6.3 \text{ km}$$

$$52° + 88° + C = 180° \Rightarrow C = 40°$$

$$\frac{c}{\sin 40°} = \frac{8}{\sin 88°} \Rightarrow c = \frac{8\sin 40°}{\sin 88°} = 5.1 \text{ km}$$

total distance $= 6.3 + 8 + 5.1 = 19.4$ km